Lecture 7

Chapter 2: Dynamic Optimization

Part II: Infinite Horizon

5/25, 2023

General Finite-Horizon Model

- There are $T + 1$ periods: $t = 0,1,2, \ldots, T$.
- You should go back to the two-period model (lecture 6) whenever you are stuck.
- The household's budget in period t is
$$y_t = c_t + s_t$$
- Asset accumulates according to
$$\begin{aligned} a_{t+1} &= R(a_t + s_t) \\ &= R(a_t + y_t - c_t) \end{aligned}$$
- Divide both sides to obtain
$$a_t + y_t = c_t + \frac{1}{R} a_{t+1}$$

General Finite-Horizon Model

- In period 0,

$$a_0 + y_0 = c_0 + \frac{1}{R} a_1$$

- Substitute the period-1 constraint into above to eliminate a_1:

$$a_0 + y_0 + \frac{y_1}{R} = c_0 + \frac{c_1}{R} + \frac{a_2}{R^2}$$

- Substitute the period-2 constraint into above to eliminate a_2:

$$a_0 + y_0 + \frac{y_1}{R} + \frac{y_2}{R^2} = c_0 + \frac{c_1}{R} + \frac{c_2}{R^2} + \frac{a_3}{R^3}$$

General Finite-Horizon Model

- Continue this for T times to finally obtain

$$a_0 + \sum_{t=0}^{T} \frac{y_t}{R^t} = \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^{T+1}}$$

- As in our previous class, we obtain the No-Ponzi-Game condition as

$$a_{T+1} \geq 0$$

General Finite-Horizon Model

- The household's problem is written as

$$\max_{\{c_t\}_{t=0}^{T}, \{a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1, \dots T$$

$$a_{T+1} \geq 0$$

- Remember that the initial wealth a_0 cannot be chosen. Likewise, in any period t, the household can only choose a_{t+1}.

General Finite-Horizon Model

- Let Λ_t be the multipliers. Then the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \Lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right]$$
$$+ \Lambda_{T+1} a_{T+1}$$

- Note that there are $T+1$ budget constraints.
- What are the FOCs?

General Finite-Horizon Model

- FOCs are:

$$c_t : \beta^t u'(c_t) - \Lambda_t = 0$$

$$a_{t+1} : -\Lambda_t \frac{1}{R} + \Lambda_{t+1} = 0$$

$$\Lambda_t : a_t + y_t = c_t + \frac{1}{R} a_{t+1}$$

$$\text{KKT} : a_{T+1} \geq 0, \Lambda_{T+1} \geq 0, \Lambda_{T+1} a_{T+1} = 0$$

- Go back to lecture 6 whenever necessary.
- For the same reasoning as the one we studied last week, we obtain $a_{T+1} = 0$ from the KKT condition.

General Finite-Horizon Model

- Eliminate the multipliers to obtain

$$\text{Euler equation}: \frac{u'(c_t)}{\beta u'(c_{t+1})} = R$$

$$\text{Budget constraint}: a_0 + \sum_{t=0}^{T} \frac{y_t}{R^t} = \sum_{t=0}^{T} \frac{c_t}{R^t}$$

- There are $T + 1$ equations in $T + 1$ unknowns.
- We can solve the system just as in the two-period model.

Example

- Let us explicitly solve the model.
- Utility function is $u(c) = \ln c$.
- Further, we assume $a_0 = 0, T = 30$, and $\beta R = 1$.
- Let $\beta = 0.99$.
- Income satisfies

$$y_t = \begin{cases} 10 \text{ for } t = 0, \dots 10 \\ 20 \text{ for } t = 11, \dots 20 \\ 0 \text{ for } t = 21, \dots 30 \end{cases}$$

Example

- The Euler equation (with $\beta R = 1$) implies

$$c_{t+1} = c_t = c$$

- The intertemporal budget constraint implies

$$\sum_{t=0}^{30} \frac{y_t}{R^t} = c \sum_{t=0}^{30} \frac{1}{R^t} \Longrightarrow c = \frac{\sum_{t=0}^{30} \beta^t y_t}{\sum_{t=0}^{30} \beta^t}$$

- Thus,

$$c = \frac{\sum_{t=0}^{10} 0.99^t 10 + \sum_{t=11}^{20} 0.99^t 20 + 0}{\sum_{t=0}^{30} 0.99^t} = \frac{275.9}{26.8}$$

$$= 10.3$$

Example

- The budget constraint for $t = 0$ implies

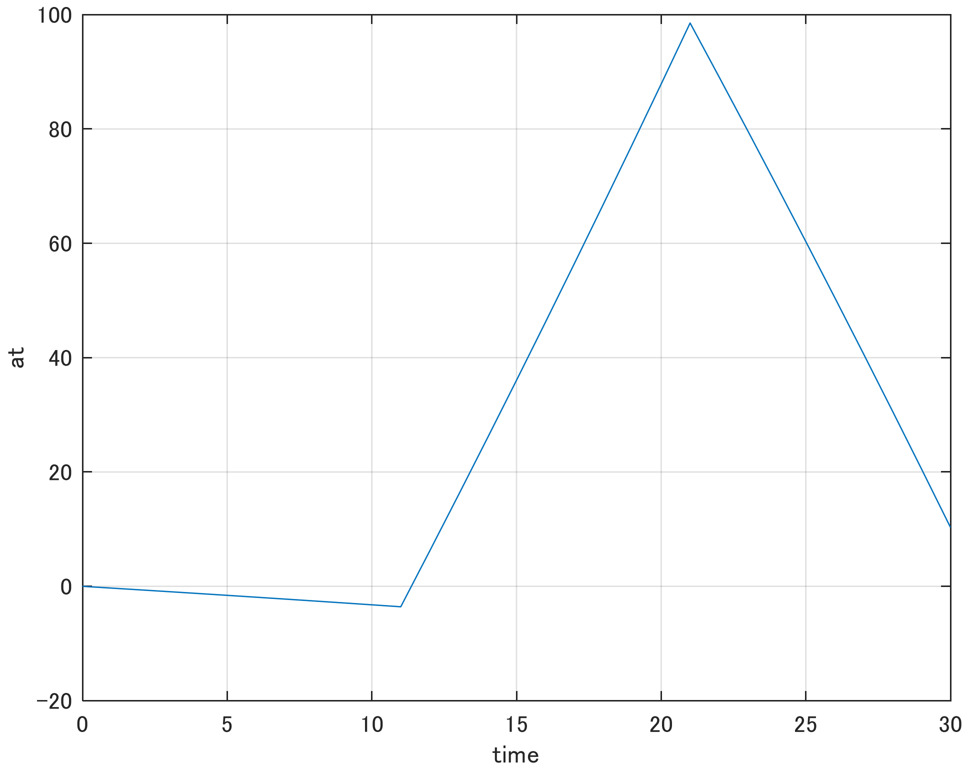
$$\underbrace{a_0}_{0} + \underbrace{y_0}_{10} = \underbrace{c_0}_{10.3} + \underbrace{\beta}_{0.99} a_1$$

- Thus,

$$a_1 = \frac{a_0 + y_0 - c_0}{\beta} = \frac{10 - 10.3}{0.99} = -0.31$$

- This process will generate a sequence of a_{t+1}.
- Matlab (Octave) code is available for download at NUCT.
  - File name is “Finite.m”

Example



Infinite-Horizon Model

- Almost every macro model has **infinite horizon**.
- The household's problem is written as

$$\max_{\{c_t\}_{t=0}^{\infty},\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1,\ldots$$

- There is no terminal period.

Infinite-Horizon Model

- The intertemporal budget constraint for the general finite-horizon model is

$$a_0 + \sum_{t=0}^{T} \frac{y_t}{R^t} = \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{a_{T+1}}{R^{T+1}}$$

- Let $T \to \infty$ to obtain

$$a_0 + \sum_{t=0}^{\infty} \frac{y_t}{R^t} = \sum_{t=0}^{\infty} \frac{c_t}{R^t} + \lim_{t\to\infty} \frac{a_t}{R^t}$$

- Is it optimal to choose $\lim_{t\to\infty} \frac{a_t}{R^t}$ to be positive?

Infinite-Horizon Model

Proposition: Optimal allocation satisfies

$$\lim_{t\to\infty} \frac{a_t}{R^t} = 0$$

Proof)

Let $\{c_0^*, c_1^*, \ldots\}, \{a_1^*, a_2^*, \ldots\}$ denote the optimal allocation. Suppose that the optimal allocation satisfies $\lim_{t\to\infty} \frac{a_t^*}{R^t} = \varepsilon > 0$.

(Our strategy is to show that this statement is contradictory.)

Proof) Continued

- Consider an alternative allocation
$$\{c_0^* + \varepsilon, c_1^*, c_2^*, \dots\}, \{a_1', a_2', \dots\}$$
- Consumption path of this allocation is exactly the same as the optimal allocation except for $t = 0$.
  - The first-period consumption $c_0' = c_0^* + \varepsilon$ is increased by $\varepsilon = \lim_{t\to\infty} \frac{a_t^*}{R^t}$. Then back to the optimal path from $t = 1$
  - Such a deviation should not be feasible.
- In period $t = 0$, the budget constraint implies
$$a_0 + y_0 = c_0^* + \varepsilon + \frac{1}{R} a_1'$$

Proof) Continued

- Thus,

$$
\begin{aligned}
a_1' &= R(a_0 + y_0 - c_0^* - \varepsilon) \\
&= R(a_0 + y_0 - c_0^*) - R\varepsilon \\
&= a_1^* - R\varepsilon
\end{aligned}
$$

- Similarly, in period 1, the budget constraint implies

$$
a_1' + y_1 = c_1^* + \frac{1}{R} a_2'
$$

- Note that in period $t = 1$, consumption is c_1^*.
- However, a_2 cannot be a_2^* because a_1' is smaller than a_1^*. In fact, $a_1' = a_1^* - R\varepsilon$.

Proof) Continued

- From the period-1 budget constraint,

$$\begin{aligned} a_2' &= R(a_1' + y_1 - c_1^*) \\ &= R(a_1^* - R\varepsilon + y_1 - c_1^*) \\ &= a_2^* - R^2\varepsilon \end{aligned}$$

- Repeating this process, we obtain

$$a_t' = a_t^* - R^t\varepsilon$$

- Thus,

$$\begin{aligned} \lim_{t\to\infty}\frac{a_t'}{R^t} = \lim_{t\to\infty}\frac{a_t^* - R^t\varepsilon}{R^t} &= \lim_{t\to\infty}\frac{a_t^*}{R^t} - \varepsilon \\ &= \varepsilon - \varepsilon = 0 \end{aligned}$$

Proof) Continued

- This means that the initial increase in consumption by $\varepsilon = \lim_{t\to\infty} \frac{a_t^*}{R^t}$ was feasible after all.
- The alternative allocation c_t' does not violate any budget constraint, yet this is strictly better than the "optimal allocation".
- This is a contradiction.
- Thus, $\lim_{t\to\infty} \frac{a_t}{R^t} = 0$ must hold at the optimal allocation, as claimed.
- End of the proof.

Infinite-Horizon Model

- Let Λ_t be the multipliers. Then the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \Lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right]$$

- There is an infinity of budget constraints.
- What are the FOCs?

Infinite-Horizon Model

- FOCs are straightforward:

$$c_t : \beta^t u'(c_t) - \Lambda_t = 0 \text{ for } t = 0,1,2,\ldots$$

$$a_{t+1} : -\Lambda_t \frac{1}{R} + \Lambda_{t+1} = 0 \text{ for } t = 0,1,2,\ldots$$

$$\Lambda_t : a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1,2,\ldots$$

- The second FOC, $\Lambda_{t+1} = \left(\frac{1}{R}\right)\Lambda_t$, is a simple difference equation, and the solution is

$$\Lambda_t = \left(\frac{1}{R}\right)^t \Lambda_0 = \frac{1}{R^t}\Lambda_0$$

Infinite-Horizon Model

- Consider $\Lambda_t = \frac{1}{R^t}\Lambda_0$.
- Substitute $\Lambda_0 = u'(c_0)$ (FOC at $t = 0$) into above:

$$\Lambda_t = \frac{1}{R^t}u'(c_0)$$

- Thus,

$$\lim_{t\to\infty}\frac{a_t}{R^t} = 0 \Leftrightarrow \lim_{t\to\infty}\frac{\Lambda_t a_t}{u'(c_0)} = 0$$

- Because $u'(c_0) > 0$,

$$\lim_{t\to\infty}\Lambda_t a_t = 0$$

Transversality Condition

- Thus, $\lim_{t\to\infty} \frac{a_t}{R^t} = 0$ is equivalent to $\lim_{t\to\infty} \Lambda_t a_t = 0$.
- Also, notice that the Lagrangian for the finite-horizon problem:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \Lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right]$$
$$+\Lambda_{T+1} a_{T+1}$$

- If we take the limit of KKT condition as $T \to \infty$, then we obtain the same condition, $\lim_{t\to\infty} \Lambda_t a_t = 0$.
- $\lim_{t\to\infty} \Lambda_t a_t = 0$ is known as the **transversality condition**.

Infinite-Horizon Model

- **Theorem**: The solution to an infinite-horizon optimization problem is given by the sequence satisfying:
  1. The budget constraints,
  2. The Euler equations,
  3. The transversality condition (TVC).

Infinite-Horizon Model

- Given the theorem, let us redo the problem

$$\max_{\{c_t\}_{t=0}^{\infty},\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0 \dots$$

- The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \Lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right]$$

Infinite-Horizon Model

- FOCs are :

$$c_t : \beta^t u'(c_t) - \Lambda_t = 0 \text{ for } t = 0,1,2,\ldots$$

$$a_{t+1} : -\Lambda_t \frac{1}{R} + \Lambda_{t+1} = 0 \text{ for } t = 0,1,2,\ldots$$

$$\Lambda_t : a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1,2,\ldots$$

$$\text{TVC} : \lim_{t \to \infty} \Lambda_t a_t = 0$$

Infinite-Horizon Model

- Thus, the optimal allocation satisfies:

1. The budget constraints:

$$a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1,2,\ldots$$

2. The Euler equation:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = R \text{ for } t = 0,1,2,\ldots$$

3. The transversality condition:

$$\lim_{t \to \infty} \Lambda_t a_t = 0$$

Infinite-Horizon Model

- Notice that the first-order conditions are given by a system of nonlinear difference equations.
- Thus, we must solve a system of difference equations to find the optimal allocation.

Current-Value Form

- Define $\lambda_t = \frac{\Lambda_t}{\beta^t}$.
- Then, we can rewrite the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \Lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right]$$

$$= \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[a_t + y_t - c_t - \frac{1}{R} a_{t+1} \right] \right\}$$

- The latter form of the Lagrangian is referred to as the **current-value form**.

Current-Value Form

- FOCs from the current value form are :

$$c_t : u'(c_t) - \lambda_t = 0 \text{ for } t = 0,1,2,\ldots$$

$$a_{t+1} : -\lambda_t \frac{1}{R} + \beta\lambda_{t+1} = 0 \text{ for } t = 0,1,2,\ldots$$

$$\lambda_t : a_t + y_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0,1,2,\ldots$$

$$\text{TVC} : \lim_{t\to\infty} \beta^t \lambda_t a_t = 0$$

- FOCs from the current-value form are <u>autonomous difference equations</u>.

Example

- Consider the following infinite-horizon problem:

$$\max_{\{c_t\}_{t=0}^{\infty},\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\ln c_t - \frac{h_t^{1+\mu}}{1+\mu} \right)$$

subject to $a_0 = 0$ and

$$a_t + w_t h_t = c_t + \frac{1}{R} a_{t+1} \text{ for } t = 0 \ldots$$

- $\mu > 0$ is a parameter.
- The sequence of wage rates $\{w_t\}_{t=0}^{\infty}$ is exogeneous.
- Assume $\beta R = 1$.

Example

a) Find the optimal consumption.
b) Find the optimal labor supply.
c) Compute the wage elasticity of labor supply. That is,

$$\frac{\partial h_t}{\partial w_t}\frac{w_t}{h_t}$$

d) Compute the wage elasticity of labor supply by keeping the Lagrange multiplier constant. This elasticity is often called the **Frisch elasticity**.