Lecture 6

Chapter 2: Dynamic Optimization

Part I: Introduction to Intertemporal Choice

5/18, 2023

Motivation

- In undergraduate macro, we used a model like
$$C = aY + b$$
  - C is consumption
  - Y is income
  - a is a parameter satisfying $0 < a < 1$
  - b is a parameter
- In graduate macro, we seek an infinite sequence of C_t by solving a utility-maximization problem.
- Today, we shall focus on a two-period model.
  - We only need to find out C_0 and C_1.

Two-Period Economy

- There are two periods, $t = 0,1$.
- Consider a household representing the economy.
- Abstracting many things, the budget constraint in period $t = 0$ is

$$\underbrace{y_0}_{\text{Income}} = \underbrace{c_0}_{\text{Consumption}} + \underbrace{s_0}_{\text{Saving}}$$

- **Saving** is a flow concept. Its stock counterpart is **asset** or **wealth**.

Two-Period Economy

- Suppose that this household has a_0 units of consumption goods as endowment.
- Let $r > 0$ denote the interest rate.
- Then, the asset at the beginning of period $t = 1$ is
$$a_1 = (1 + r)(a_0 + s_0)$$
- To shorten the expression, let $R = 1 + r$ to write
$$a_1 = R(a_0 + s_0)$$
- We eliminate s_0 to obtain
$$a_1 = R(a_0 + y_0 - c_0)$$

Two-Period Economy

- Divide both sides by R to obtain

$$a_0 + y_0 = c_0 + \frac{1}{R} a_1$$

- According to this expression, you can spend both your initial wealth a_0 and your income (such as wage income) to make consumption c_0 and to purchase financial asset a_1 at the price $1/R$.
- If $s_0 < 0$, then you are borrowing (flow concept).
- If $a_1 < 0$, then you are in debt (stock concept).

Two-Period Economy

- In period $t = 1$, the budget constraint is
$$y_1 = c_1 + s_1$$
- Asset for the next period is
$$a_2 = R(a_1 + s_1) = R(a_1 + y_1 - c_1)$$
- Divide both sides by R to obtain
$$a_1 + y_1 = c_1 + \frac{a_2}{R}$$
- Use it to eliminate a_1 from the equation on page 5:
$$a_0 + y_0 = c_0 + \frac{1}{R}\left[c_1 + \frac{a_2}{R} - a_1 - y_1\right]$$

Two-Period Economy

- Arrange terms to obtain

$$\underbrace{a_0 + y_0 + \frac{y_1}{R}}_{\text{Life-time wealth}} = c_0 + \frac{c_1}{R} + \frac{a_2}{R^2}$$

- This is called the **intertemporal budget constraint**.
- Suppose that you can choose $a_2 = -\infty$.
  - What does it mean?
  - Stop for the moment and think before you move on.

Two-Period Economy

- As stated on page 5, a negative wealth means that you are indebted.
  - Thus, $a_2 < 0$ means that you die with the debt. You do not repay the debt.
  - Thus, $a_2 = -\infty$ means that you can spend as much as you want by borrowing and not replaying.
- However, no rational individual will allow you to do so. Thus, we need an additional constraint:

$$a_2 \geq 0$$

- This constraint is called the **No-Ponzi-Game (NPG) condition**, named after Charles Ponzi (1982-1949).

Two-Period Economy

- Let us now turn to the household's preferences.
- In microeconomics, when there are two goods, x and y, utility function is generally defined as $U(x, y)$.
- The basic idea here is that we treat c_0 and c_1 as <u>different commodities</u>.
  - An apple you consume today and the same apple you consume tomorrow are different commodities because you need to wait until tomorrow.

Two-Period Economy

- In general, utility from consuming c_0 units and c_1 units are given by $U(c_0, c_1)$.
- It is common to assume U to be **time separable**:
$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$
  - This means that in each period t, you evaluate the utility from consumption. Then you add them up by discounting, using the **discount factor** $\beta < 1$.
- You can also consider the **discount rate** ρ such that
$$\beta = \frac{1}{1+\rho}$$

Two-Period Economy

- Consider a time-separable utility:
$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$
- **Instantaneous utility function** $u(c)$ is assumed to be increasing and concave:
$$u'(c) > 0 > u''(c)$$
- Further, we assume the **Inada condition**:
$$\lim_{c \to 0} u'(c) = \infty$$
$$\lim_{c \to \infty} u'(c) = 0$$

Two-Period Economy

- We now have all the expressions we need.
- The household's problem is written as

$$\max_{c_0,c_1,a_1,a_2} u(c_0) + \beta u(c_1)$$

subject to

$$a_0 + y_0 = c_0 + \frac{1}{R} a_1$$

$$a_1 + y_1 = c_1 + \frac{1}{R} a_2$$

$$a_2 \geq 0$$

Two-Period Economy

- Let $\Lambda_0, \Lambda_1, \Lambda_2$ be the Lagrange multipliers. Then the Lagrangian is

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \Lambda_0 \left[a_0 + y_0 - c_0 - \frac{1}{R} a_1\right] + \Lambda_1 \left[a_1 + y_1 - c_1 - \frac{1}{R} a_2\right] + \Lambda_2 a_2$$

- What are the first-order conditions?

Two-Period Economy

- The FOCs with respect to consumption are easy:

$$c_0 : u'(c_0) - \Lambda_0 = 0$$
$$c_1 : \beta u'(c_1) - \Lambda_1 = 0$$

- To calculate the FOC with respect to a_1, observe

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \Lambda_0 \left[a_0 + y_0 - c_0 - \frac{1}{R} a_1 \right] + \Lambda_1 \left[a_1 + y_1 - c_1 - \frac{1}{R} a_2 \right] + \Lambda_2 a_2$$

Two-Period Economy

- Thus, the FOC with respect to a_1 is

$$a_1 : -\Lambda_0 \frac{1}{R} + \Lambda_1 = 0$$

- Similarly, we obtain

$$a_2 : -\Lambda_1 \frac{1}{R} + \Lambda_2 = 0$$

Two-Period Economy

- The FOCs with respect to the multipliers give us the budget constraints:

$$a_0 + y_0 = c_0 + \frac{1}{R} a_1$$

$$a_1 + y_1 = c_1 + \frac{1}{R} a_2$$

Two-Period Economy

- Finally, we need to deal with the inequality constraint $a_2 \geq 0$.
- The intuition tells us that you should choose $a_2 = 0$ because $a_2 > 0$ means you lend money before you die. This seems irrational.
- In the real world, we observe $a_2 > 0$ for many reasons:
  - Uncertainty: You do not usually know when to die.
  - Bequest: If you have children, you are more than happy to leave money to them.

Two-Period Economy

- The condition associated with the inequality constraint is
$$\text{KKT}: a_2 \geq 0, \Lambda_2 \geq 0, \Lambda_2 a_2 = 0$$
- This is called the **Karush-Kuhn-Tucker condition**, formally known as the **Kuhn-Tucker condition**.
  - I know the inequality constraint $a_2 \geq 0$ is not required for solving this model as it is evident that $a_2 = 0$ is optimal.
  - The purpose here is to extend the idea to understand the more general condition, the **transversality condition** for the infinite-horizon problem.

Karush-Kuhn-Tucker

- For now, consider the following problem:

$$\max_{x} f(x)$$

subject to

$$x \geq 0$$

- Suppose that function $f(x)$ has one peak.
- Then, without the inequality constraint, the solution (=peak) is easily found by solving:

$$f'(x) = 0$$

- Let the peak be x^*.
- The problem is, there is no guarantee that x^* is found in the region $x > 0$.

Karush-Kuhn-Tucker

- Let λ denote the Lagrange multiplier. Then,

$$\mathcal{L} = f(x) + \lambda x$$

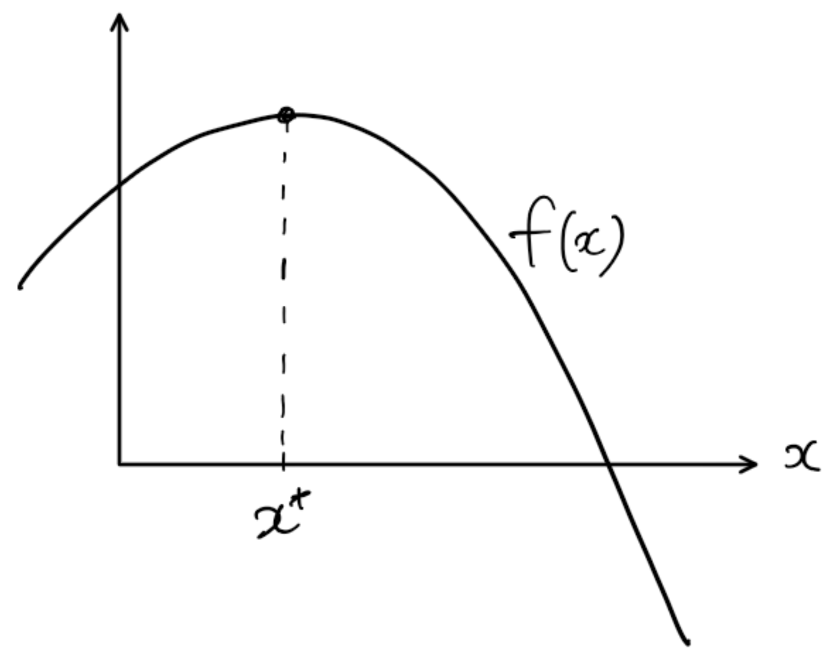
- The FOCs are

$$x : f'(x) + \lambda = 0$$
$$KKT : x \geq 0, \lambda \geq 0, \lambda x = 0$$

- Because $\lambda x = 0$, either λ or x must be zero.
- There are two cases to consider:
  - Case 1: $\lambda = 0$ and $x^* > 0$
  - Case 2: $\lambda > 0$ and $x^* = 0$

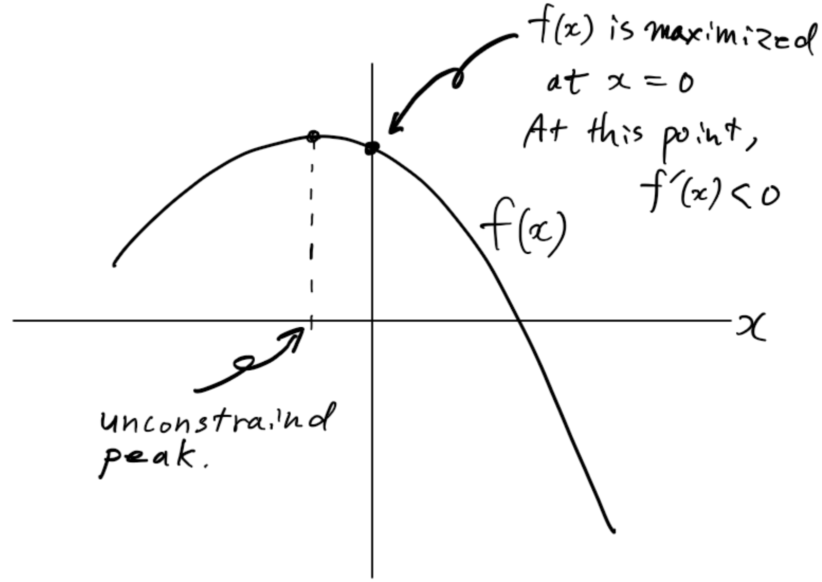
Karush-Kuhn-Tucker

- Consider Case 1: $\lambda = 0$ and $x^* > 0$
- The FOCs simply require $f'(x) = 0$.
- In this case, the peak x^* is found in the region $x \geq 0$.



Karush-Kuhn-Tucker

- Consider Case 2: $\lambda > 0$ and $x^* = 0$
- The FOCs imply
$$f'(x) = -\lambda < 0$$
- In this case, the peak x^* is found <u>at the edge</u> of the region $x \geq 0$.



Two-Period Economy

- Let us go back to our two-period model.

$$\text{KKT} : a_2 \geq 0, \Lambda_2 \geq 0, \Lambda_2 a_2 = 0$$

- We shall consider two cases:
  - Case 1: $\Lambda_2 = 0$ and $a_2 > 0$
  - Case 2: $\Lambda_2 > 0$ and $a_2 = 0$
- In Case 1, one of the FOCs implies

$$-\Lambda_1 \frac{1}{R} + \Lambda_2 = 0 \Rightarrow -\Lambda_1 \frac{1}{R} = 0 \Rightarrow \Lambda_1 = 0$$

- However, another FOC implies

$$\beta u'(c_1) - \Lambda_1 = 0 \Rightarrow \beta u'(c_1) = 0$$

Two-Period Economy

- Consider
$$\beta u'(c_1) = 0$$
- Because of the Inada condition $\lim_{c \to \infty} u'(c) = 0$, for a finite level of consumption, we can never find the solution to $u'(c_1) = 0$.
- Thus, there is no solution in Case 1.

Two-Period Economy

- Now consider case 2: $\Lambda_2 > 0$ and $a_2 = 0$
- The FOCs are summarized by

$$c_0 : u'(c_0) - \Lambda_0 = 0$$
$$c_1 : \beta u'(c_1) - \Lambda_1 = 0$$
$$a_1 : -\Lambda_0 \frac{1}{R} + \Lambda_1 = 0$$
$$a_2 : -\Lambda_1 \frac{1}{R} + \Lambda_2 = 0$$
$$a_0 + y_0 = c_0 + \frac{1}{R} a_1$$
$$a_1 + y_1 = c_1$$

Two-Period Economy

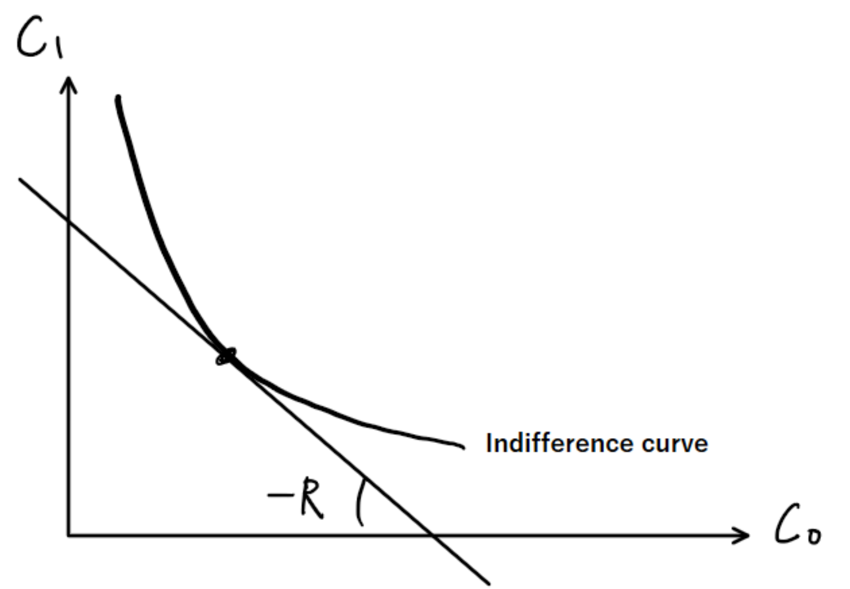
- Eliminate the multipliers to obtain

  Euler equation : $\dfrac{u'(c_0)}{\beta u'(c_1)} = R$

  Budget constraint : $a_0 + y_0 + \dfrac{y_1}{R} = c_0 + \dfrac{c_1}{R}$
- Notice that this is a system of nonlinear equations.
- There are two equations in two unknowns c_0 and c_1.

Two-Period Economy

- We can draw a diagram just as in the standard undergraduate micro.
- The Euler equation tells us that the slope of the indifference curve and the slope of the budget constraint must be the same.



Two-Period Economy

- To explicitly solve the equations, we need to specify $u(c)$. The most commonly-used functional form is
$$u(c) = \ln c$$
- Economics only considers natural logarithm, so we may also write $\log c$.
- $\ln c$ satisfies all assumptions we impose such as $u' > 0 > u''$ and the Inada condition.
- More importantly, $u'(c) = 1/c$. This greatly simplifies the analysis.

Two-Period Economy

- Another functional form employed in many macro models is **CRRA** form:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- CRRA means constant relative risk aversion.
- The marginal utility is $u'(c) = c^{-\sigma}$.
- This function contains $\ln c$ as a special case when $\sigma = 1$.
  - To exactly derive $\ln c$, we need to start with $\frac{c^{1-\sigma}-1}{1-\sigma}$. But this form is not very popular.

Two-Period Economy

- Let us solve the equations with $u(c) = \ln c$:

$$\text{Euler equation}: \frac{1/c_0}{\beta/c_1} = R \Leftrightarrow c_1 = \beta R c_0$$

$$\text{Budget constraint}: a_0 + y_0 + \frac{y_1}{R} = c_0 + \frac{c_1}{R}$$

- We can easily solve them and obtain:

$$c_0 = \frac{a_0 + y_0 + \frac{y_1}{R}}{1+\beta}$$

$$c_1 = \frac{\beta}{1+\beta} R \left[a_0 + y_0 + \frac{y_1}{R} \right]$$