Lecture 5

Chapter 1: Difference Equations

Part V: Nonlinear Systems

5/11, 2023

Nonlinear Systems

- Consider:

$$x_{t+1} = x_t - \alpha x_t y_t$$
$$y_{t+1} = y_t + \alpha x_t y_t - \beta y_t$$

- This is an example of a **nonlinear system**.
  - We shall come back to this system later.
- We can generally describe the system as

$$\begin{cases} x_{t+1} = f(x_t, y_t) \\ y_{t+1} = g(x_t, y_t) \end{cases}$$

Nonlinear Systems

- Consider a two-dimensional **nonlinear system**:

$$\begin{cases} x_{t+1} = f(x_t, y_t) \\ y_{t+1} = g(x_t, y_t) \end{cases}$$

- Steps:
  1. Find all steady states.
  2. Pick one steady state.
  3. Linearize the system around the steady state.
  4. Study the linearized system.
     - Referred to as a **local analysis**.

Nonlinear Systems

- Let (x, y) denote a steady state of the system. Then,

$$\begin{cases} x = f(x, y) \\ y = g(x, y) \end{cases}$$

  - There may be two or more steady states.
  - The existence of a steady state is not guaranteed.
  - Thus, if your model is nonlinear, then it is critical that you <u>count the number of steady states</u>.
- In what follows, we assume that there is at least one steady state.
  - Notations I use are x, x_{SS}, $\bar{x}$. They all mean the same.

Nonlinear Systems

- Remember that, for x close enough to a point $\bar{x}$, we can linearly approximate $f(x)$ at $\bar{x}$ as
$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$$
- Now consider $f(x, y)$. For (x, y) close enough to a point $(\bar{x}, \bar{y})$, we can linearly approximate $f(x, y)$ at $(\bar{x}, \bar{y})$ as
$$f(x, y) = f(\bar{x}, \bar{y}) + \frac{\partial f(\bar{x}, \bar{y})}{\partial x}(x - \bar{x}) + \frac{\partial f(\bar{x}, \bar{y})}{\partial y}(y - \bar{y})$$
- In what follows, we denote
$$\frac{\partial f(x, y)}{\partial x} = f_1, \frac{\partial f(x, y)}{\partial y} = f_2$$

Nonlinear Systems

- Thus, we can linearize the two-dimensional nonlinear system around (x, y) as:

$$\begin{cases} x_{t+1} = x + f_1(x,y)(x_t - x) + f_2(x,y)(y_t - y) \\ y_{t+1} = y + g_1(x,y)(x_t - x) + g_2(x,y)(y_t - y) \end{cases}$$

- In matrix form,

$$\begin{pmatrix} x_{t+1} - x \\ y_{t+1} - y \end{pmatrix} = \begin{pmatrix} f_1(x,y) & f_2(x,y) \\ g_1(x,y) & g_2(x,y) \end{pmatrix} \begin{pmatrix} x_t - x \\ y_t - y \end{pmatrix}$$

- More compactly,

$$z_{t+1} = J z_t$$

- J is called the **Jacobian matrix** of the system.

Nonlinear Systems

- As before, we want more speed.
- Let us totally differentiate the system and evaluate each coefficient at its steady-state value to obtain

$$\begin{cases} dx_{t+1} = f_1 dx_t + f_2 dy_t \\ dy_{t+1} = g_1 dx_t + g_2 dy_t \end{cases}$$

- In matrix form,

$$\begin{pmatrix} dx_{t+1} \\ dy_{t+1} \end{pmatrix} = J \begin{pmatrix} dx_t \\ dy_t \end{pmatrix}$$

- More compactly,

$$z_{t+1} = J z_t$$

Nonlinear Systems

- In many macroeconomic models, explicit expressions ($x_{t+1} = \cdots$) are not available.
- Thus, the most general description of a two-dimensional nonlinear system is in **implicit form**:

$$\begin{cases} F(x_{t+1}, y_{t+1}, x_t, y_t) = 0 \\ G(x_{t+1}, y_{t+1}, x_t, y_t) = 0 \end{cases}$$

- A steady state (x, y) satisfies

$$\begin{cases} F(x, y, x, y) = 0 \\ G(x, y, x, y) = 0 \end{cases}$$

Nonlinear Systems

- Linearize the system around (x, y) to obtain
$$\begin{cases} F_1 dx_{t+1} + F_2 dy_{t+1} + F_3 dx_t + F_4 dy_t = 0 \\ G_1 dx_{t+1} + G_2 dy_{t+1} + G_3 dx_t + G_4 dy_t = 0 \end{cases}$$
- In matrix form,
$$\begin{pmatrix} F_1 & F_2 \\ G_1 & G_2 \end{pmatrix} \begin{pmatrix} dx_{t+1} \\ dy_{t+1} \end{pmatrix} = \begin{pmatrix} -F_3 & -F_4 \\ -G_3 & -G_4 \end{pmatrix} \begin{pmatrix} dx_t \\ dy_t \end{pmatrix}$$
- More compactly, let $z_t = (dx_t, dy_t)'$ to write
$$Bz_{t+1} = Cz_t$$
- If B is invertible, then
$$z_{t+1} = B^{-1} C z_t = A z_t$$

Stability

- We are now ready to formally discuss the stability of a dynamical system.
- Definition: A steady state $\bar{x}$ is **globally (asymptotically) stable** if $\lim_{t \to \infty} x_t = \bar{x}$ for any x_0.
- Definition: A steady state $\bar{x}$ is **locally stable** if $\lim_{t \to \infty} x_t = \bar{x}$ for any x_0 in the neighborhood of $\bar{x}$ (i.e., x_0 such that $|x_0 - \bar{x}| < \varepsilon$ for some $\varepsilon > 0$).
- For any linear system, these two stability concepts are the same.

Stability

- Consider a first-order linear scalar equation:
$$x_{t+1} = ax_t + b$$
- Steady state $\bar{x}$ is globally stable if and only if
$$|a| < 1$$
- To see this, consider the sequence $\{x_t\}_{t=0}^{\infty}$ generated by the equation, which is given by
$$x_t = \bar{x} + a^t(x_0 - \bar{x})$$
- Evidently, for any initial condition x_0,
$$\lim_{t \to \infty} x_t = \bar{x}$$
holds if and only if $|a| < 1$.

Stability

- Consider a first-order nonlinear scalar equation
$$x_{t+1} = f(x_t)$$
- Steady state $\bar{x}$ is locally stable if and only if
$$|f'(\bar{x})| < 1$$
- To see this, consider the linearized equation
$$\hat{x}_{t+1} = f'(\bar{x})\hat{x}_t$$
- The solution, or the sequence $\{\hat{x}_t\}_{t=0}^{\infty}$, is given by
$$\hat{x}_t = [f'(\bar{x})]^t \hat{x}_0$$
- Given $\{\hat{x}_t\}_{t=0}^{\infty}$, we know $\{x_t\}_{t=0}^{\infty}$ by $x_t = \hat{x}_t + \bar{x}$.

Stability

- Because we obtain the linearized equation by considering the neighborhood of one particular steady state $\bar{x}$, our initial condition x_0 must be chosen from the neighborhood of of $\bar{x}$.
- In other words, we can no longer talk about the global stability of a nonlinear system by looking at its linearized equation.

Stability

- Consider the sequence $\{x_t\}_{t=0}^{\infty}$ generated by the linearized equation, which is given by
$$x_t = \bar{x} + [f'(\bar{x})]^t (x_0 - \bar{x})$$
- Evidently, for any initial condition x_0 in the neighborhood of $\bar{x}$, we can show that
$$\lim_{t \to \infty} x_t = \bar{x}$$
holds if and only if $|f'(\bar{x})| < 1$.
- This is the condition for **local stability**.

Stability

- Consider a higher-dimensional linear system:

$$\begin{pmatrix} x_{t+1}^1 \\ \vdots \\ x_{t+1}^n \end{pmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} x_t^1 \\ \vdots \\ x_t^n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- More compactly, $x_{t+1} = Ax_t + b$
- Steady state is $x = -(A - I)^{-1}b$
- The general solutions is

$$x_t = x + \sum_{i=1}^{n} c_i \lambda_i^t P_i$$

Stability

- **Theorem:** Suppose all eigenvalues of A have moduli different from 1. Then the unique steady state of the system, x, is globally (asymptotically) stable if and only if all the eigenvalues of A have moduli strictly smaller than 1, and unstable if at least one eigenvalue has modulus strictly larger than 1.
- If you have a higher-dimensional **nonlinear** system, then you should study its Jacobian matrix J for each steady state.

Stability

- An important caveat is that we have not yet considered any **equilibrium concept**.
- It is very, very important to know at this point that a steady state being stable does **not** necessarily mean that the equilibrium at that point is stable.
  - For example, a saddle point is unstable as a dynamical system. However, we will learn that the saddle path is the unique equilibrium path.
  - This happens because we will consider forward-looking rational individuals. Remember: Econ $\neq$ Physics.

Applications

Solow Growth Model

- A good example of a nonlinear system is the Solow model. Let K_t and N_t denote capital and labor used in production. Then, a country's aggregate output Y_t (i.e., real GDP) is determined by $Y_t = F(K_t, N_t)$.
- We adopt the **Neoclassical assumptions**:
  - $F_1 > 0, F_2 > 0, F_{11} < 0, F_{22} < 0$
  - Inada conditions:
    - $\lim_{K \to 0} F_1 = \infty, \lim_{K \to \infty} F_1 = 0$
    - $\lim_{N \to 0} F_2 = \infty, \lim_{N \to \infty} F_2 = 0$
  - Constant Returns to Scale

Solow Growth Model

- Most widely used specification is Cobb-Douglas:
$$Y_t = AK_t^{\alpha} N_t^{1-\alpha}$$
- α satisfies $1 < \alpha < 1$
  - Often referred to as the capital share.
  - $\alpha = 1/3$ in many countries in many era.
- A is referred to as Total Factor Productivity (TFP).

Solow Growth Model

- The key driving force of economic growth in the Solow model is capital formation over time.
- Stock of capital evolves according to the following difference equation:

$$K_{t+1} = K_t + I_t - \delta K_t$$

- I_t is capital formation (investment) in t.
- δ is the depreciation rate ($0 < \delta < 1$).

Solow Growth Model

- One household represents the entire economy (the representative household).
- Household has two roles:
    - Supplier of capital (via saving)
    - Supplier of labor
- Savings are exogenously determined by
$$S_t = sY_t$$
- s is the saving rate ($0 < s < 1$)
- Equivalently, consumption decision is exogenous:
$$C_t = (1 - s)Y_t$$

Solow Growth Model

- Household supplies labor input inelastically. In other words, labor supply is exogenous. Thus,
$$\text{Labor input} = \text{Population}$$
- Population is assumed to grow exogenously:
$$N_{t+1} = nN_t$$
- $n > 1$ is the **gross** population growth rate.
  - Also referred to as the population growth **factor**.
  - You might prefer $N_{t+1} = (1+n)N_t$ instead.
    - This is fine, too.
    - This is just a matter of taste.

Solow Growth Model

- We consider a closed economy without government.
- Without government and foreign countries, output is either consumed or invested:

$$Y_t = C_t + I_t$$

  - This is the goods market equilibrium condition.
- As usual, we can transform the goods market equilibrium condition into the capital market equilibrium condition:

$$I_t = Y_t - C_t = S_t$$

Solow Growth Model

- Model summary:

$$Y_t = AK_t^{\alpha}N_t^{1-\alpha}$$
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$S_t = sY_t$$
$$N_{t+1} = nN_t$$
$$I_t = S_t$$

- Note that the Solow model is essentially a system of nonlinear difference equations.
- 5 equations in 5 unknowns.
  - By substitutions, we can reduce the dimension.

Solow Growth Model

- We have a two-dimensional nonlinear system:

$$\begin{cases} K_{t+1} = sAK_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t \\ N_{t+1} = nN_t \end{cases}$$

- $(0,0)$ is a steady state. Not interesting, though.
- $N_{t+1} = nN_t$ implies $N_t = n^t N_0$. Thus,

$$K_{t+1} = sAK_t^{\alpha}(n^t N_0)^{1-\alpha} + (1-\delta)K_t$$

- This scalar equation is a **non-autonomous** DE.
- K_t never stop growing. Thus, $(0,0)$ is <u>the only steady state</u>.

Solow Growth Model

- Instead of searching for a steady state, we look for a balanced-growth path (BGP).
- Transform the system as follows

$$\frac{N_{t+1}}{Nt} \times \frac{K_{t+1}}{N_{t+1}} = sAK_t^{\alpha} N_t^{-\alpha} + (1-\delta)\frac{K_t}{N_t}$$

- Thus,

$$nk_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t$$

- $k_t = K_t/N_t$ is the capital-labor ratio.
- BGP is a solution to $nk = sAk^{\alpha} + (1+\delta)k$.

Solow Growth Model

- BGP (steady state in terms of k_t) satisfies
$$k(sAk^{\alpha-1} + 1 - \delta - n) = 0$$
- Thus, $k = 0$ and
$$k = \left(\frac{sA}{n + \delta - 1}\right)^{\frac{1}{1-\alpha}} \equiv k_{ss}$$
- Computing the impact of s on BGP is easy:
$$\frac{\partial k_{ss}}{\partial s} = \frac{1}{1-\alpha}\left(\frac{sA}{n + \delta - 1}\right)^{\frac{1}{1-\alpha}-1} \frac{A}{n + \delta - 1} > 0$$

Solow Growth Model

- Consider

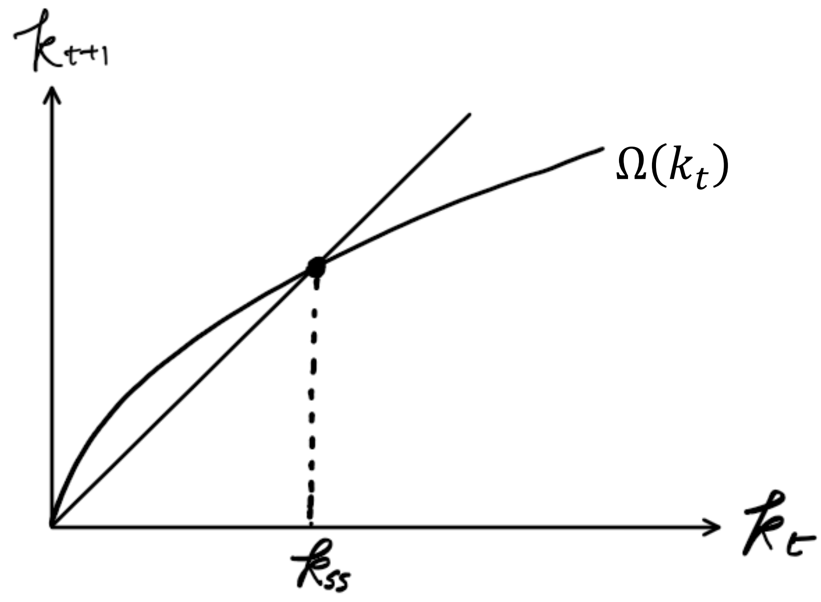
$$k_{t+1} = \frac{sAk_t^\alpha + (1-\delta)k_t}{n} \equiv \Omega(k_t)$$

- From this,

$$\Omega'(k_t) = \frac{\alpha sA}{n} k_t^{\alpha-1} + \frac{1-\delta}{n}$$
$$\Omega'(0) = \infty > 1$$
$$\Omega'(k_{ss}) = \alpha + (1-\alpha)\frac{1-\delta}{n} < 1$$

Solow Growth Model

- $\Omega'(0) = \infty > 1$ implies that the steady state at $(0,0)$ is **locally unstable**.
- $\Omega'(k_{ss}) = \alpha + (1-\alpha)\frac{1-\delta}{n} < 1$ implies that the other steady state is **locally stable**.



Solow Growth Model

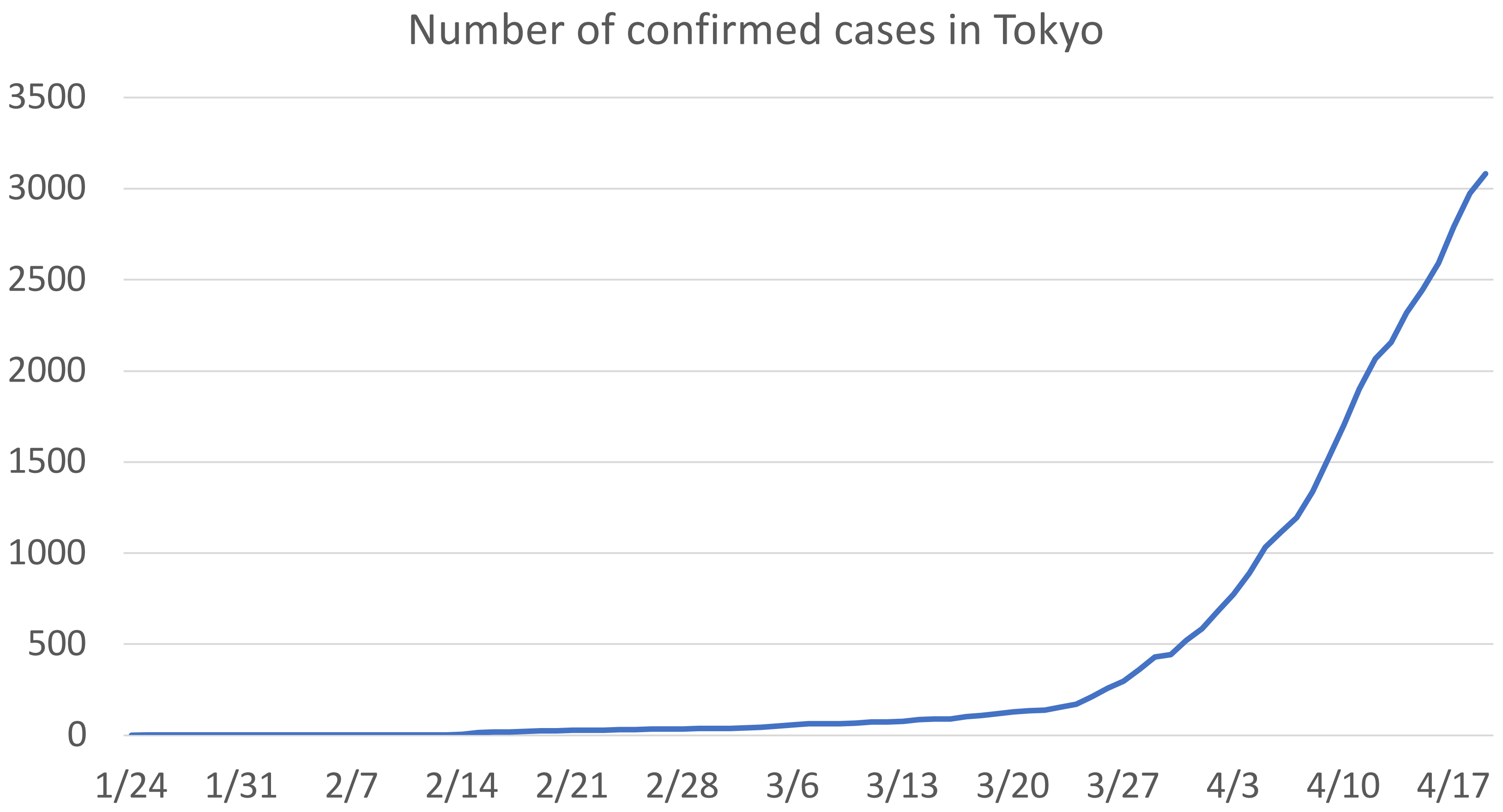
- Log-linearize the equation around k_{ss}:

$$\hat{k}_{t+1} = \left[\frac{\alpha sA}{n} k_{ss}^{\alpha-1} + \frac{1-\delta}{n}\right]\hat{k}_t$$

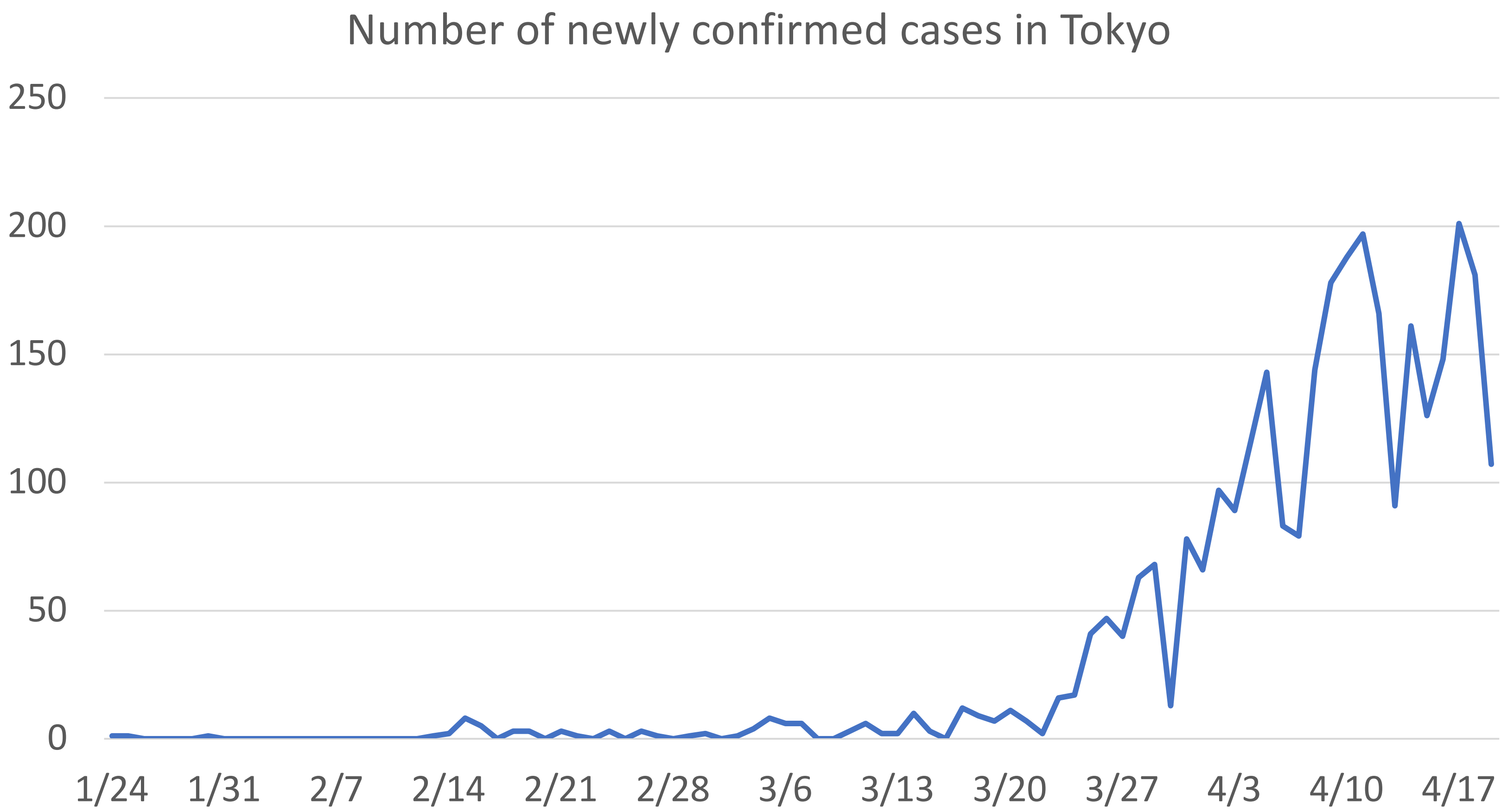
- Because k_{ss} satisfies $sAk_{ss}^{\alpha-1} + 1 - \delta - n = 0$, We can simplify the log-linearized equation as

$$\hat{k}_{t+1} = \left[\alpha + (1-\alpha)\frac{1-\delta}{n}\right]\hat{k}_t$$

Covid-19 Outbreak in 2020



Covid-19 Outbreak in 2020



SIR Model

- The Covid-19 model presented here is known as the **SIR** model.
- Individuals are classified into 3 **states**.
- Let x_t denote the number of "**s**usceptible" individuals who have not infected in date t.
- Let y_t denote the number of "**i**nfected" (and circulating) individuals in date t.
- Let z_t denote the number of "**r**emoved" individuals (either died, isolated, or recovered).

SIR Model

- Let us assume that **total number of contacts** within a day is $x_t y_t$.
- Let α denote the **infection rate**, which is assumed to be a parameter.
- Then, the number of newly infected individuals in day t is given by $\alpha x_t y_t$.
- An infected is removed at rate β, a parameter.
- Then the number of removed individuals in day t is βy_t.

SIR Model

- Total number of susceptible in date $t+1$ is
$$x_{t+1} = x_t - \alpha x_t y_t$$
- Total number of infected in date $t+1$ is
$$y_{t+1} = y_t + \alpha x_t y_t - \beta y_t$$
- Total number of removed in date $t+1$ is
$$z_{t+1} = z_t + \beta y_t$$
- Let n denote total population. Then,
$$x_t + y_t + z_t = n$$

SIR Model

- Without loss of generality, we can focus on:

$$x_{t+1} = x_t - \alpha x_t y_t$$
$$y_{t+1} = y_t + \alpha x_t y_t - \beta y_t$$

- Let us first find the steady states. Let (x, y) satisfy

$$x = x - \alpha x y$$
$$y = y + \alpha x y - \beta y$$

- They reduce to

$$xy = 0$$
$$(\alpha x - \beta) y = 0$$

- Thus, <u>any point</u> on the x-axis ($y = 0$) is a steady state.

SIR Model

- Let us study the global behavior of the model through a phase diagram:

$$x_{t+1} - x_t = -\alpha x_t y_t$$
$$y_{t+1} - y_t = \alpha x_t y_t - \beta y_t$$

- Note that x_t and y_t cannot be negative. Thus,

$$\Delta x < 0$$

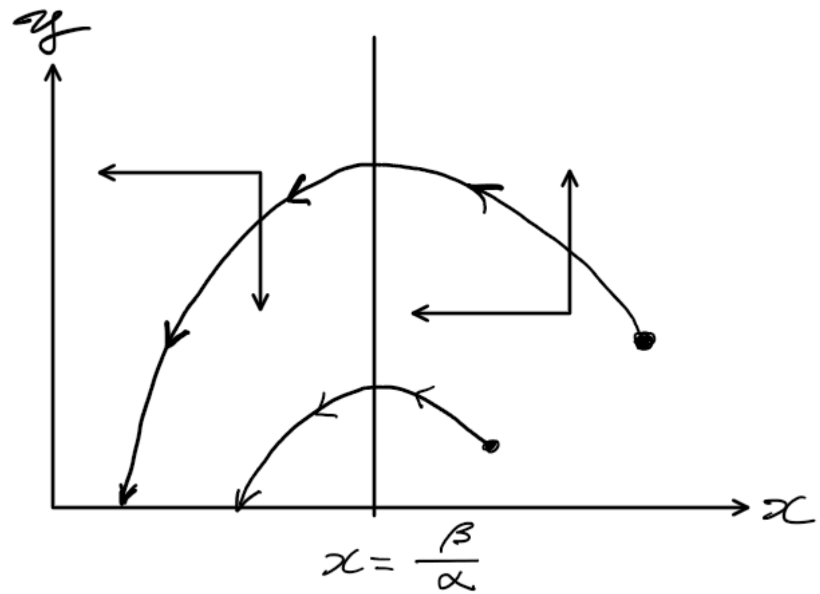
for any region.

- Similarly, for any $y_t > 0$,

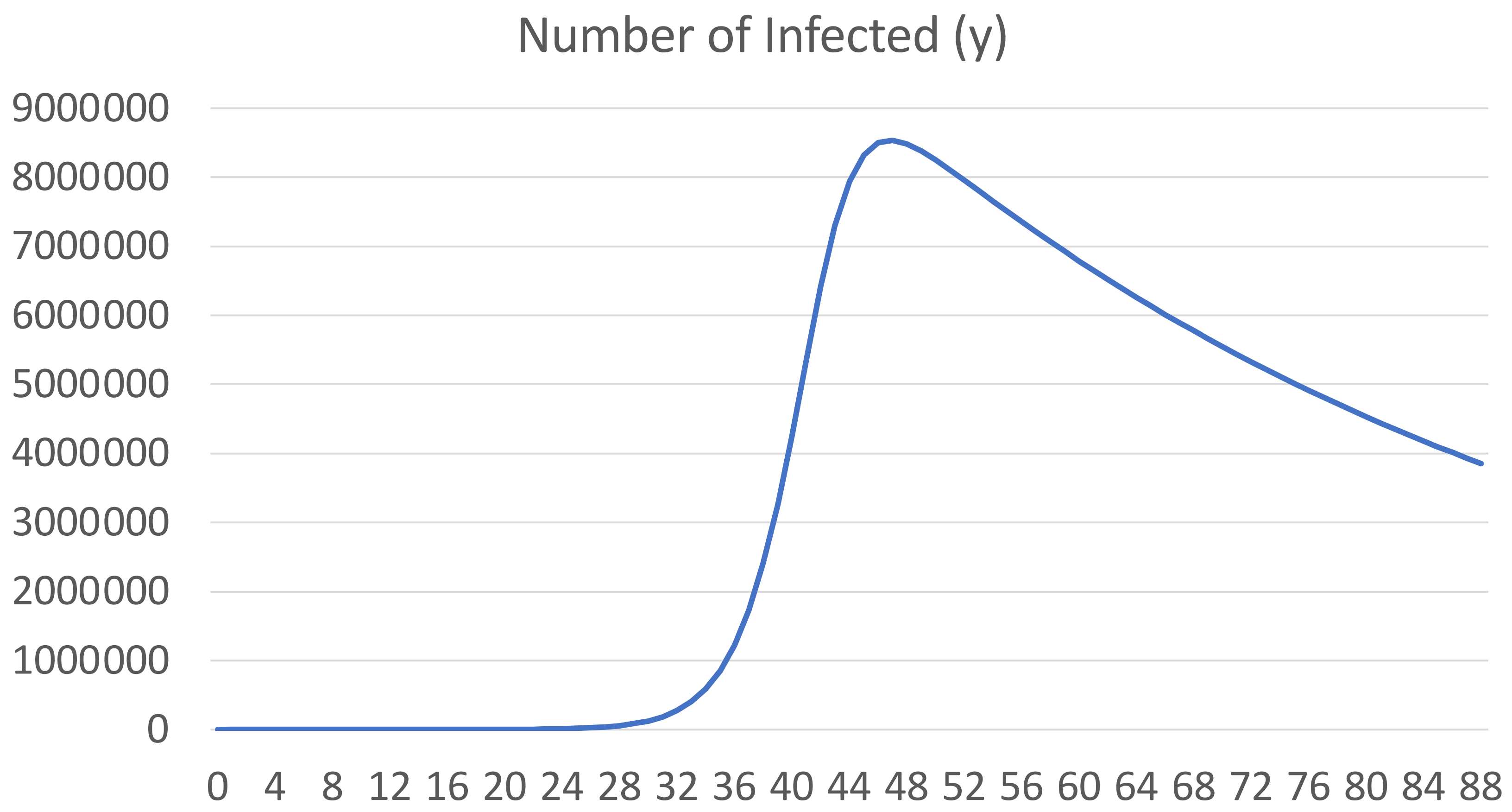
$$\Delta y > 0 \Leftrightarrow x_t > \frac{\alpha}{\beta}$$

SIR Model

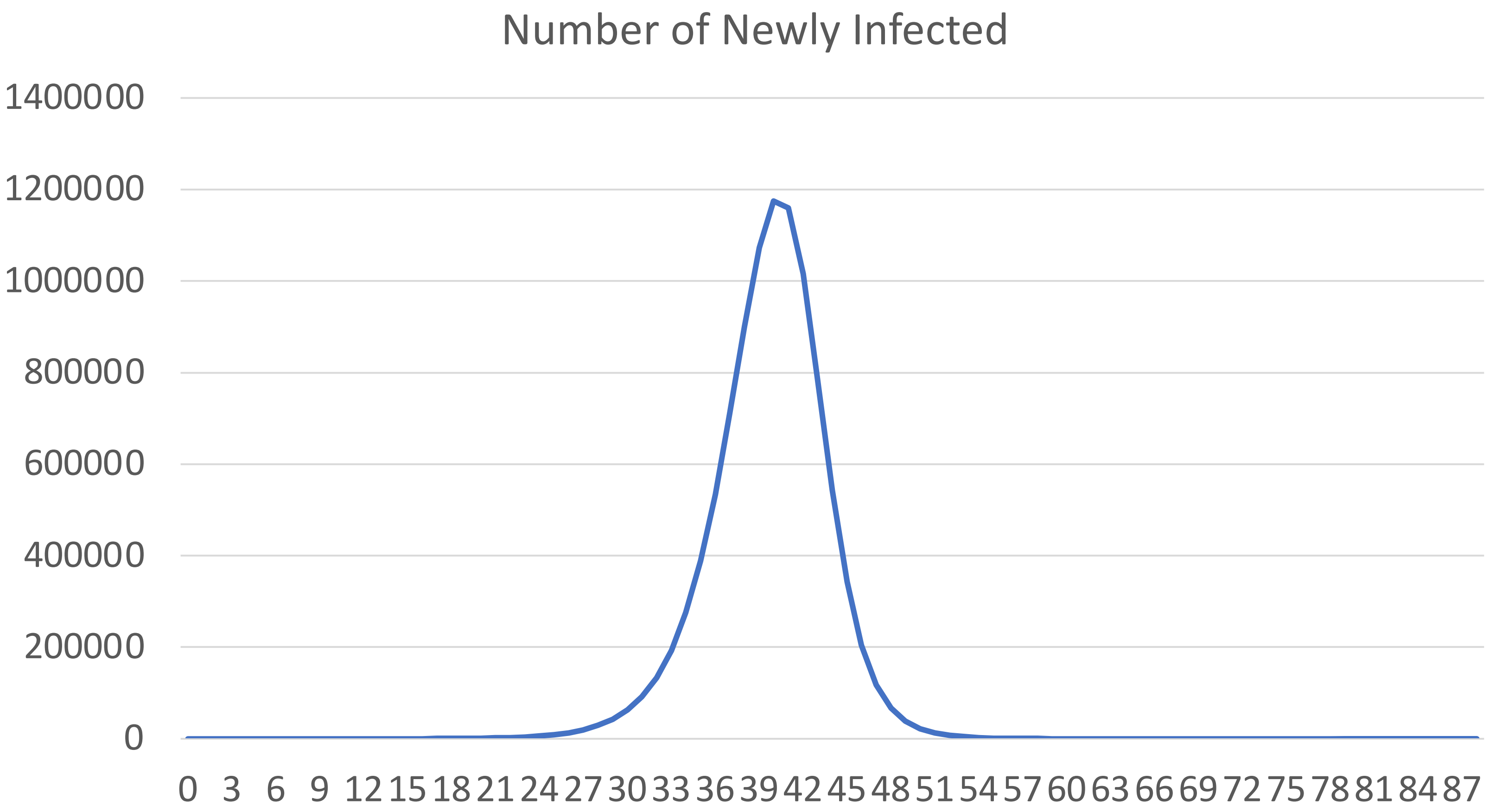
- Each initial condition leads to a distinct trajectory.
- The steady state level of y is necessarily 0.
- However, the steady state level of x takes any value, depending on the initial condition.
- Local analysis seems to be meaningless.



Simulating SIR Model

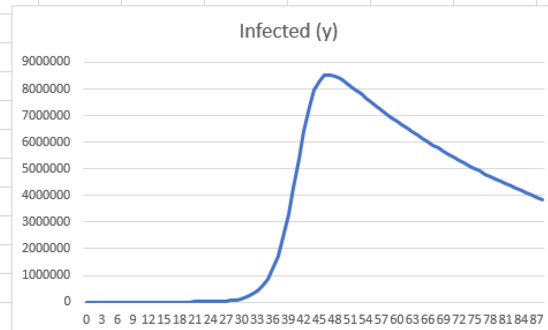


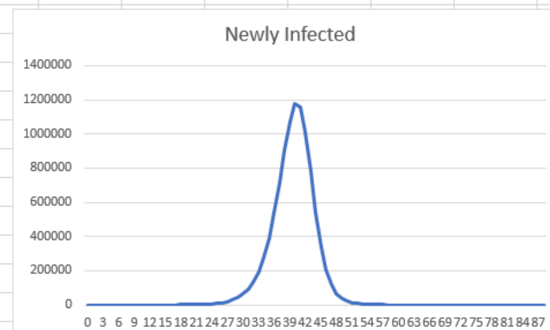
Simulating SIR Model



Results are from Excel

t	Infection rate	Removal rate	Susceptible (x)	Infected (y)	Removed (z)	Newly Infected	Removed
0	0.00000005	0.02	9999999	1	0	0.49999995	0.02
1	0.00000005	0.02	9999998.5	1.47999995	0.02	0.739999864	0.0296
2	0.00000005	0.02	9999997.76	2.190399815	0.049599999	1.095199662	0.043808
3	0.00000005	0.02	9999996.665	3.241791481	0.093407995	1.6208952	0.064836
4	0.00000005	0.02	9999995.044	4.797850851	0.158243825	2.398924237	0.095957
5	0.00000005	0.02	9999992.645	7.100818071	0.254200842	3.550406424	0.142016
6	0.00000005	0.02	9999989.095	10.50920813	0.396217203	5.254598336	0.210184
7	0.00000005	0.02	9999983.84	15.55362231	0.606401366	7.776798586	0.311072
8	0.00000005	0.02	9999976.063	23.01934845	0.917473812	11.50964667	0.460387
9	0.00000005	0.02	9999964.554	34.06860815	1.377860781	17.03424369	0.681372
10	0.00000005	0.02	9999947.519	50.42147968	2.059232944	25.21060753	1.00843
11	0.00000005	0.02	9999922.309	74.62365762	3.067662538	37.31153893	1.492473
12	0.00000005	0.02	9999884.997	110.4427234	4.56013569	55.22072664	2.208854
13	0.00000005	0.02	9999829.776	163.4545956	6.768990158	81.72590659	3.269092
14	0.00000005	0.02	9999748.051	241.9114103	10.03808207	120.9526577	4.838228
15	0.00000005	0.02	9999627.098	358.0258397	14.87631027	179.0062444	7.160517
16	0.00000005	0.02	9999448.092	529.8715673	22.03682707	264.9211616	10.59743
17	0.00000005	0.02	9999183.17	784.1952976	32.63425842	392.0656211	15.68391
18	0.00000005	0.02	9998791.105	1160.577013	48.31816437	580.2183556	23.21154
19	0.00000005	0.02	9998210.886	1717.583828	71.52970462	858.6382664	34.35168
20	0.00000005	0.02	9997352.248	2541.870418	105.8813812	1270.598697	50.83741
21	0.00000005	0.02	9996081.65	3761.631707	156.7187895	1880.078884	75.23263
22	0.00000005	0.02	9994201.571	5566.477956	231.9514237	2781.625137	111.3296
23	0.00000005	0.02	9991419.945	8236.773534	343.2809828	4114.853169	164.7355
24	0.00000005	0.02	9987305.092	12186.89123	508.0164535	6085.710043	243.7378





Further Readings

- Luenberger. *Introduction to Dynamic Systems*, John Wiley & Sons, 1979.
- Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press, 2009.
  - Chapter 2 “The Solow Growth Model”