Lecture 4

Chapter 1: Difference Equations

Part IV: Stability Analysis

5/2, 2023

Phase Diagram

- Consider

$$\begin{cases} x_{t+1} = ax_t + by_t \\ y_{t+1} = cx_t + dy_t \end{cases}$$

- We can rewrite them as

$$\begin{cases} x_{t+1} - x_t = (a-1)x_t + by_t \\ y_{t+1} - y_t = cx_t + (d-1)y_t \end{cases}$$

- In our example (as in the previous lecture),

$$\begin{cases} x_{t+1} - x_t = 1.5y_t \\ y_{t+1} - y_t = 0.5x_t - y_t \end{cases}$$

Phase Diagram

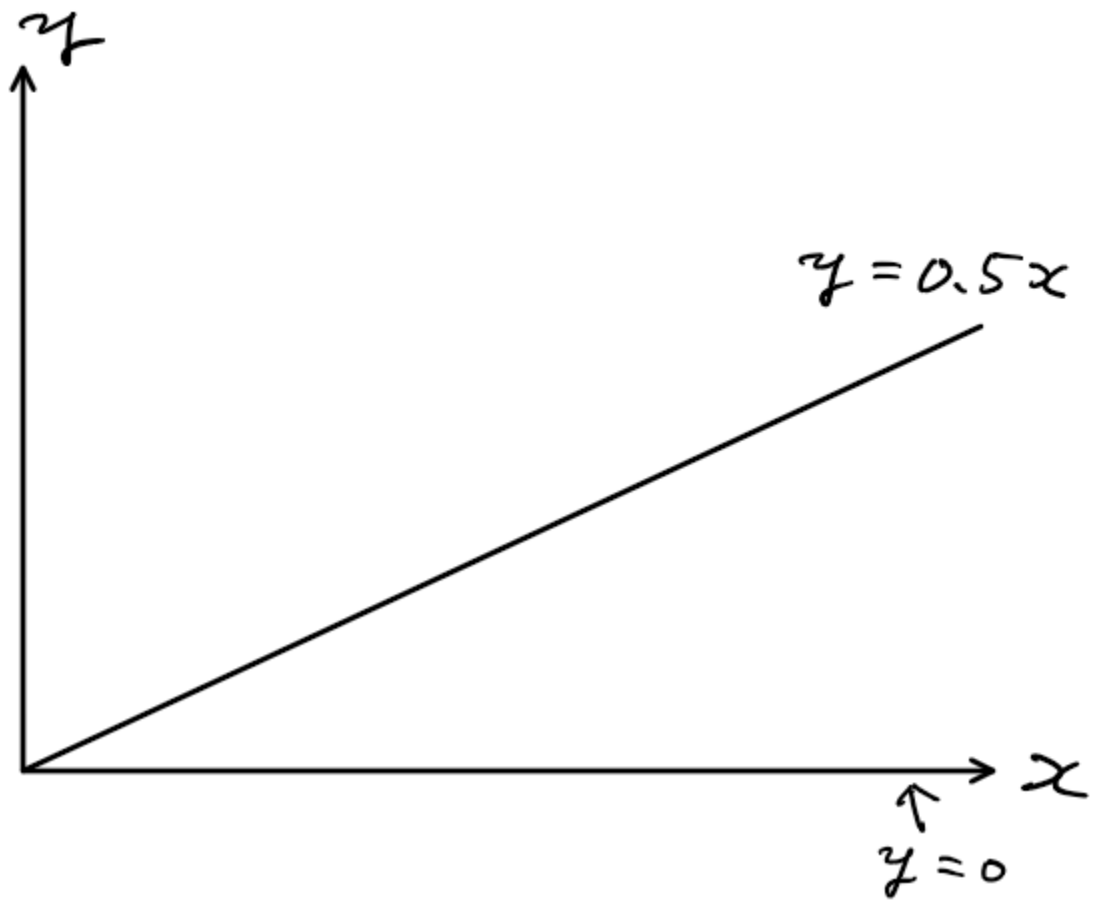
- Because $x_{t+1} - x_t$ means the change in x from t to $t+1$, we denote it by Δx.
- Thus, we obtain

$$\begin{cases} \Delta x = 1.5 y_t \\ \Delta y = 0.5 x_t - y_t \end{cases}$$

- Let us first consider the steady state, in which all variables are constant over time:

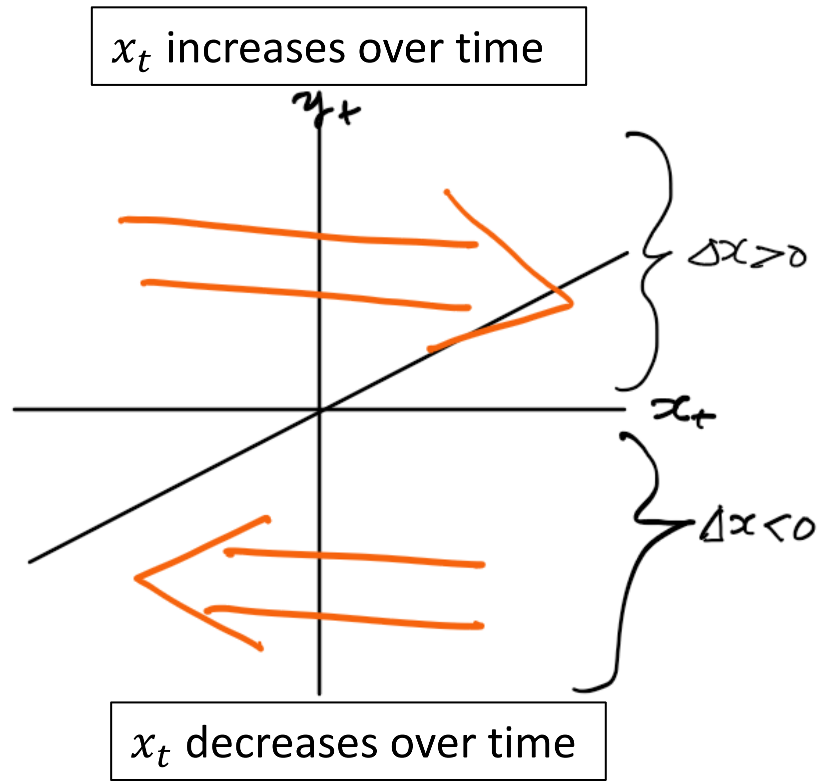
$$\begin{cases} \Delta x = 0 \\ \Delta y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ y = 0.5x \end{cases}$$

Phase Diagram



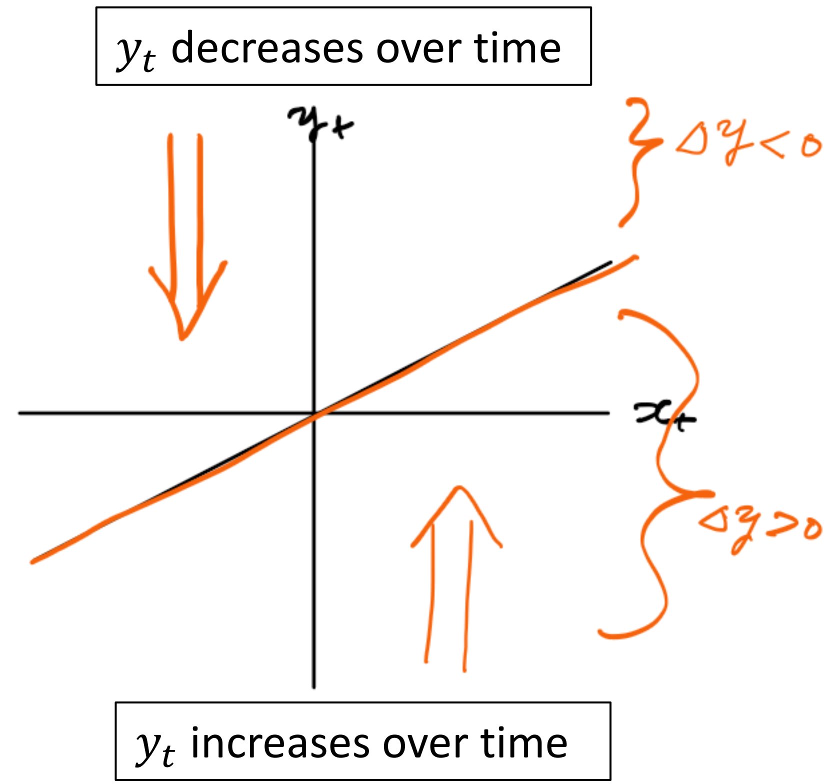
Phase Diagram

- Note that $\Delta x = 0$ is equivalent to $y = 0$.
- Likewise, consider the region in which x_t grows over time. Such a region must satisfy $\Delta x > 0$.
- $\Delta x = 1.5 y_t$ implies that $\Delta x > 0$ holds if and only if $y_t > 0$.



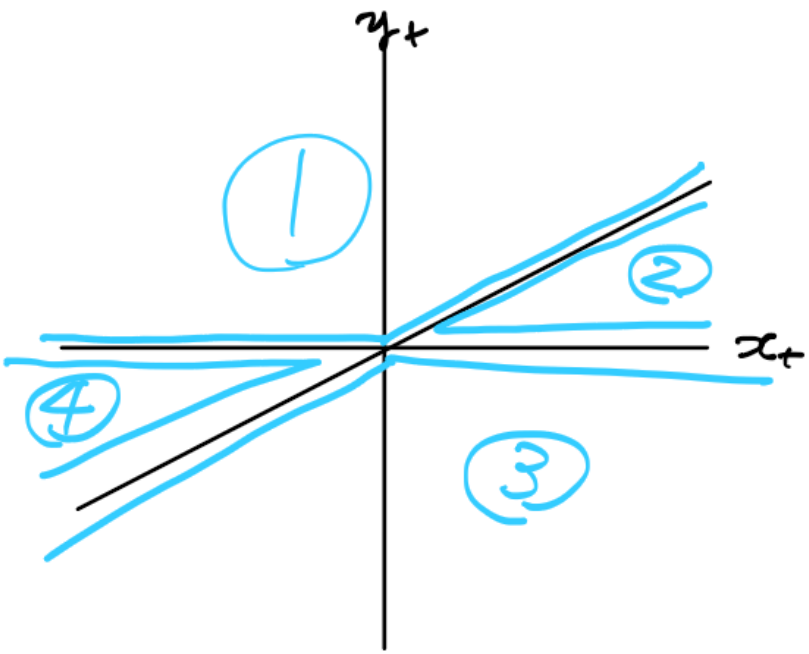
Phase Diagram

- Similarly, $\Delta y = 0$ is equivalent to $y = 0.5x$.
- Now, consider the region in which y_t <u>grows over time</u>. Such a region must satisfy $\Delta y > 0$.
- $\Delta y = 0.5x_t$ implies that $\Delta y > 0$ holds if and only if $0.5x_t > y_t$.

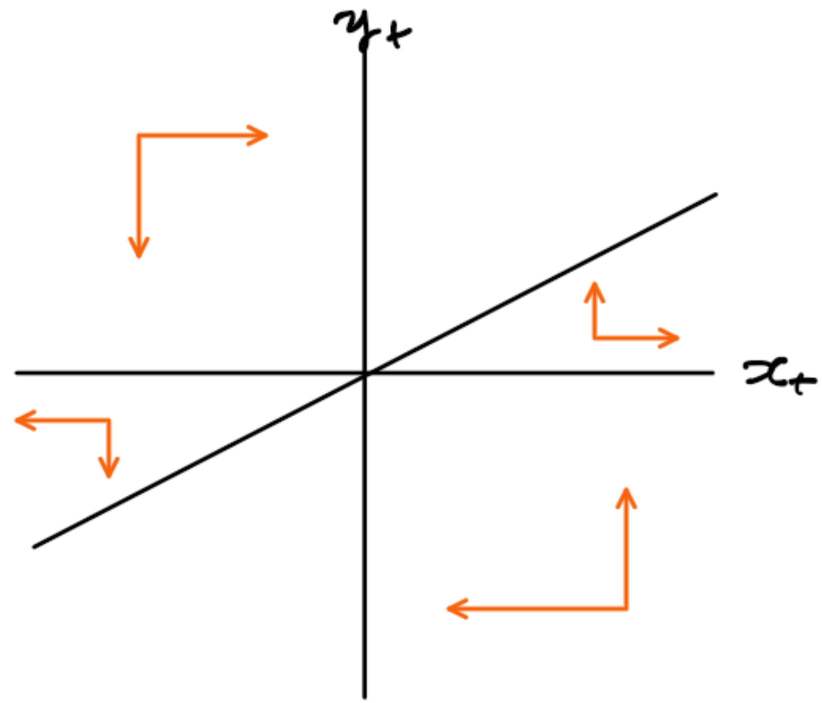


Phase Diagram

There are 4 regions

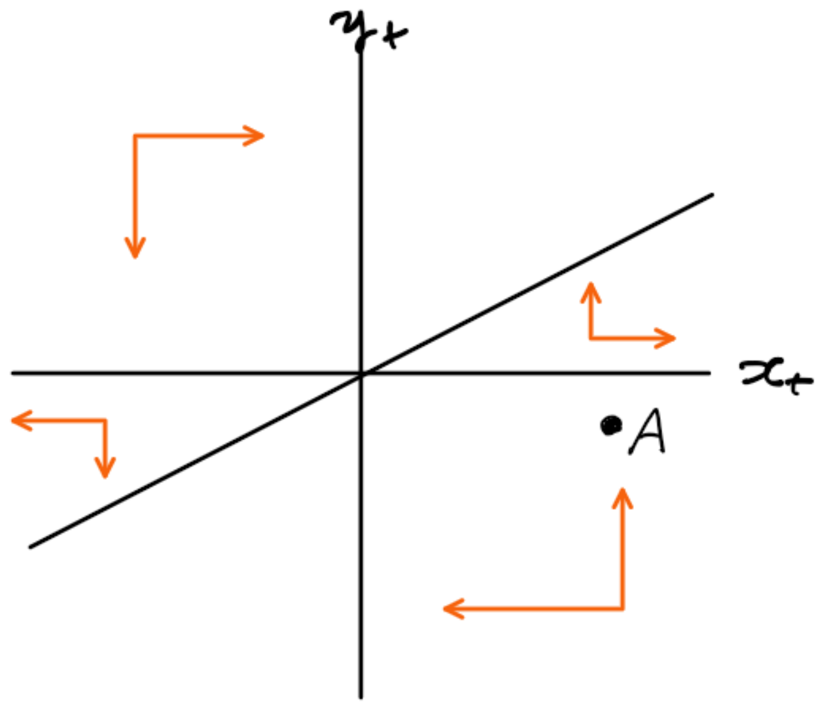


In each region, we know which direction to go



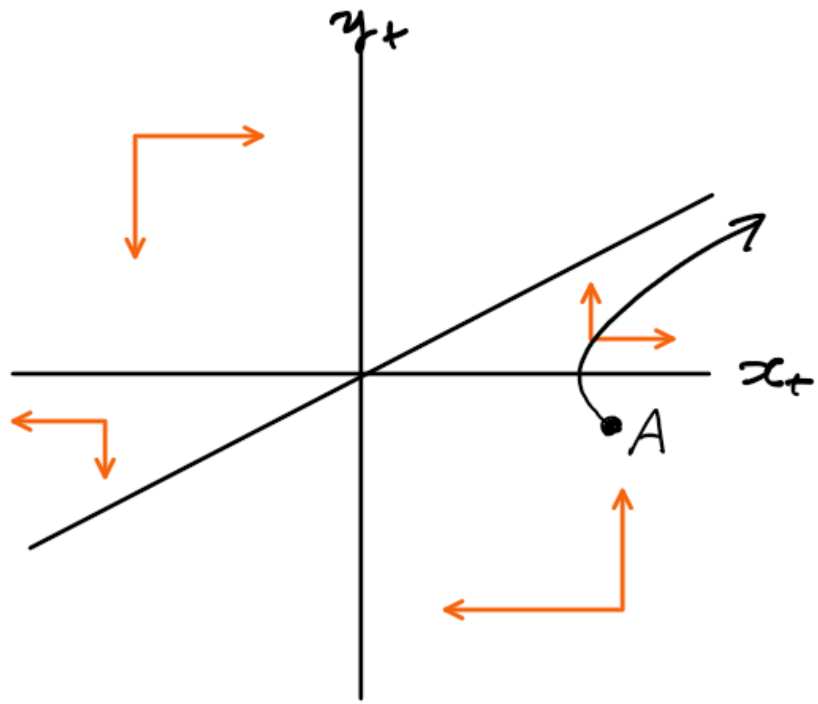
Phase Diagram

- Suppose that point A is the initial condition (x_0, y_0).
- Which way will the economy go?



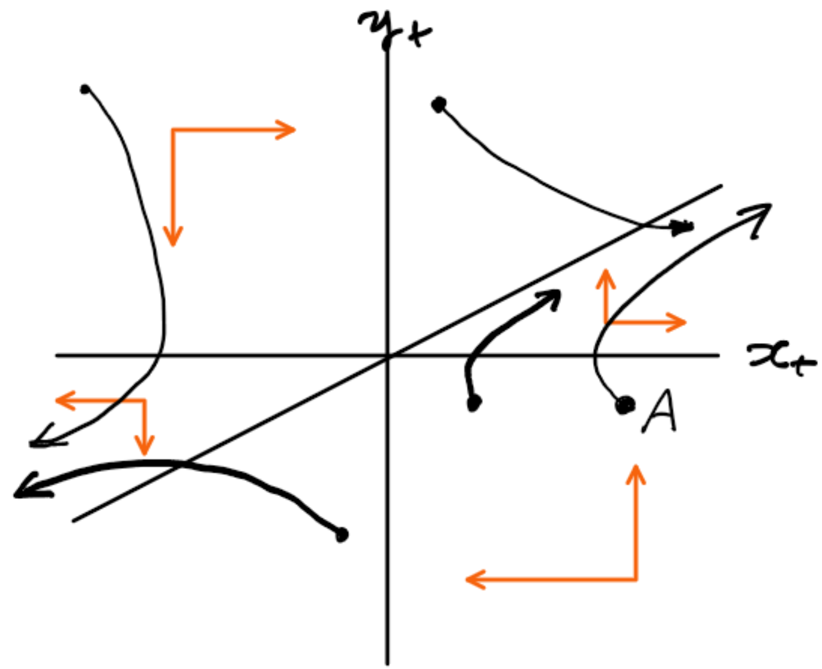
Phase Diagram

- The trajectory starting from point A is **divergent**.
- Let us consider many other points.



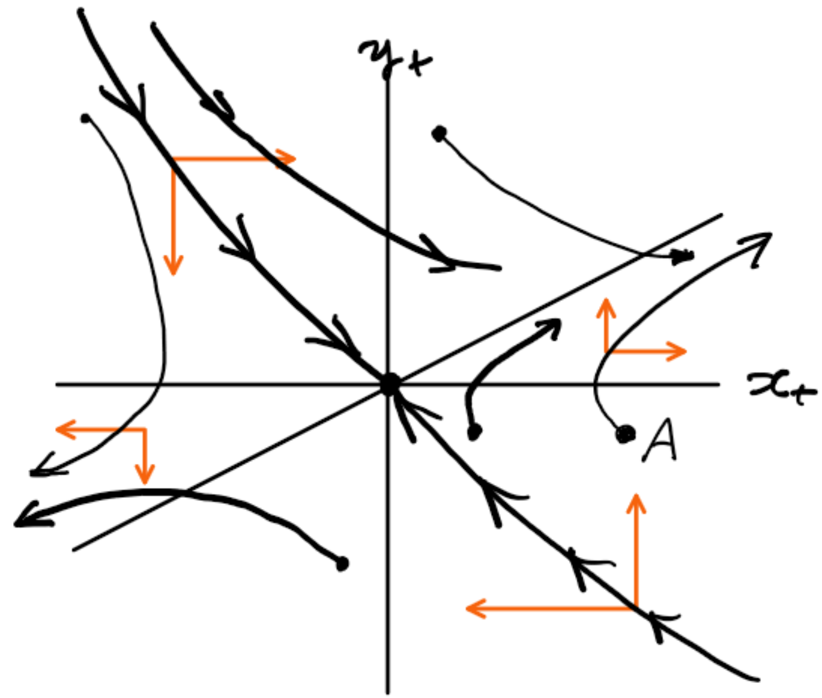
Phase Diagram

- There is an infinity of divergent (or, explosive) trajectories.
- Does it mean that the steady state (which is at the origin) cannot be reached?



Phase Diagram

- Not really.
- There is exactly one trajectory leading to the steady state.
- This trajectory is referred to as the **saddle path**.

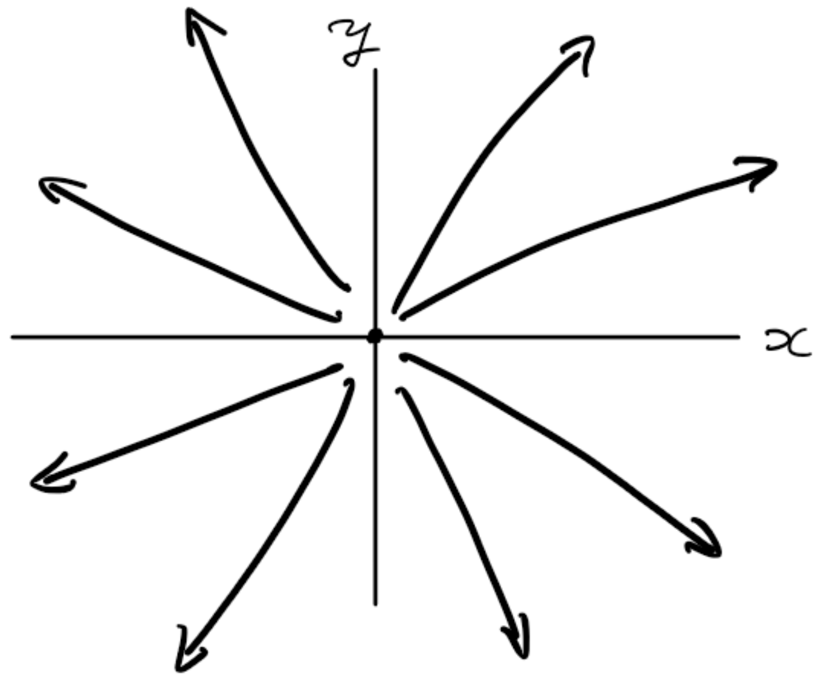


Stability of Linear Systems

- Stability of a linear system depends on the absolute values of its eigenvalues:
  1. $|\lambda_1| > 1$ and $|\lambda_2| > 1 \Longrightarrow$ Explosive (source)
  2. $|\lambda_1| < 1$ and $|\lambda_2| < 1 \Longrightarrow$ Convergent (sink)
  3. $|\lambda_1| < 1 < |\lambda_2| \Longrightarrow$ Saddle

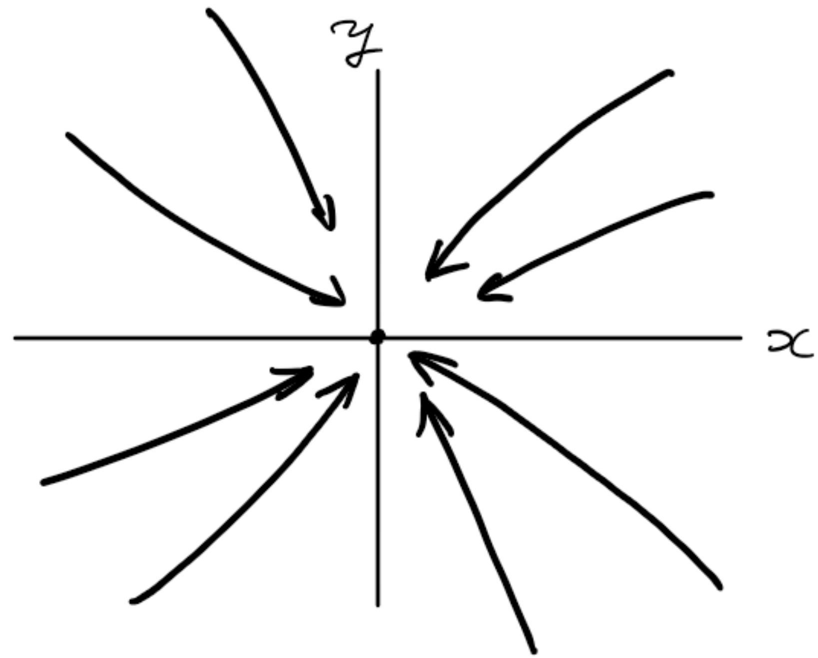
Stability of Linear Systems

- $|\lambda_1| > 1$ and $|\lambda_2| > 1$ $\Longrightarrow$ **Source**
- Remember that the general solution is
$$\begin{cases} x_t = c_1 e_{11} \lambda_1^t + c_2 e_{21} \lambda_2^t \\ y_t = c_1 e_{12} \lambda_1^t + c_2 e_{22} \lambda_2^t \end{cases}$$
- Evidently, when $|\lambda_1| > 1$ and $|\lambda_2| > 1$, both x_t and y_t are explosive.



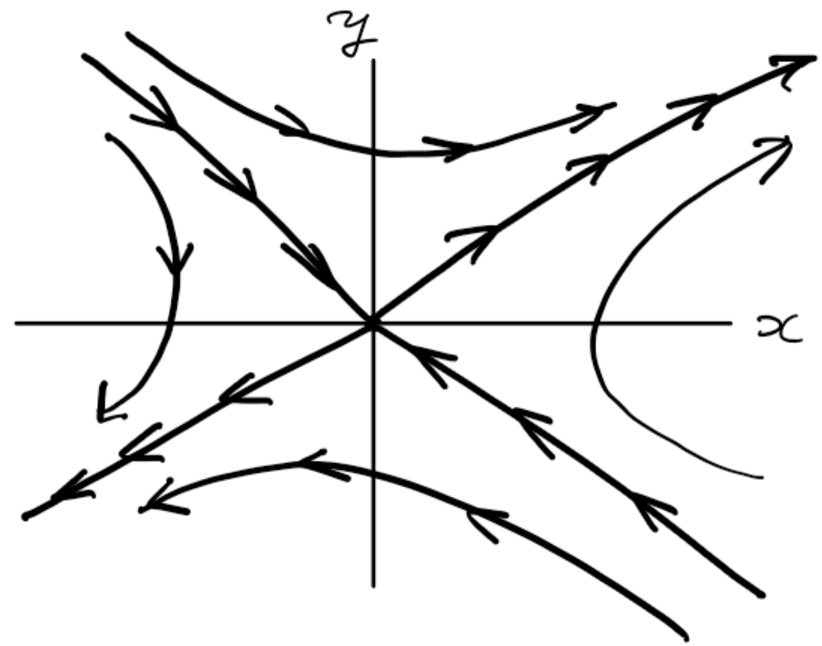
Stability of Linear Systems

- $|\lambda_1| < 1$ and $|\lambda_2| < 1$ $\Longrightarrow$ **Sink**
- Remember that the general solution is
$$\begin{cases} x_t = c_1 e_{11} \lambda_1^t + c_2 e_{21} \lambda_2^t \\ y_t = c_1 e_{12} \lambda_1^t + c_2 e_{22} \lambda_2^t \end{cases}$$
- Evidently, when $|\lambda_1| < 1$ and $|\lambda_2| < 1$, both x_t and y_t converge to the origin as $t \to \infty$.



Stability of Linear Systems

- $|\lambda_1| < 1 < |\lambda_2|$
  $\Longrightarrow$ **Saddle**
- In our example, $\lambda_1 = -0.5$ and $\lambda_2 = 1.5$.
- It will be useful to know how to compute the saddle path.



Computing the Saddle Path

- Remember that the general solution is
$$\begin{cases} x_t = c_1 e_{11} \lambda_1^t + c_2 e_{21} \lambda_2^t \\ y_t = c_1 e_{12} \lambda_1^t + c_2 e_{22} \lambda_2^t \end{cases}$$
- When $|\lambda_1| < 1 < |\lambda_2|$, as $t \to \infty$, $\lambda_1^t \to 0$ and $\lambda_2^t \to \infty$ or $-\infty$.
- Thus, whenever $c_2 \neq 0$, the system will explode.
- In other words, we should be looking for the initial condition such that $c_2 = 0$.

Computing the Saddle Path

- Imposing $c_2 = 0$, we obtain

$$\begin{cases} x_t = c_1 e_{11} \lambda_1^t \\ y_t = c_1 e_{12} \lambda_1^t \end{cases}$$

- Eliminating λ_1^t from the two equations, we obtain

$$y_t = \frac{e_{12}}{e_{11}} x_t$$

- This is the saddle path.
- In our example (see lecture 3), $e_{12} = -1$ and $e_{11} = 1$. Thus, the saddle path is $y_t = -x_t$.

Complex Eigenvalues

- Consider

$$\begin{cases} x_{t+1} = x_t + y_t \\ y_{t+1} = -9x_t + y_t \end{cases}$$

- In matrix form,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

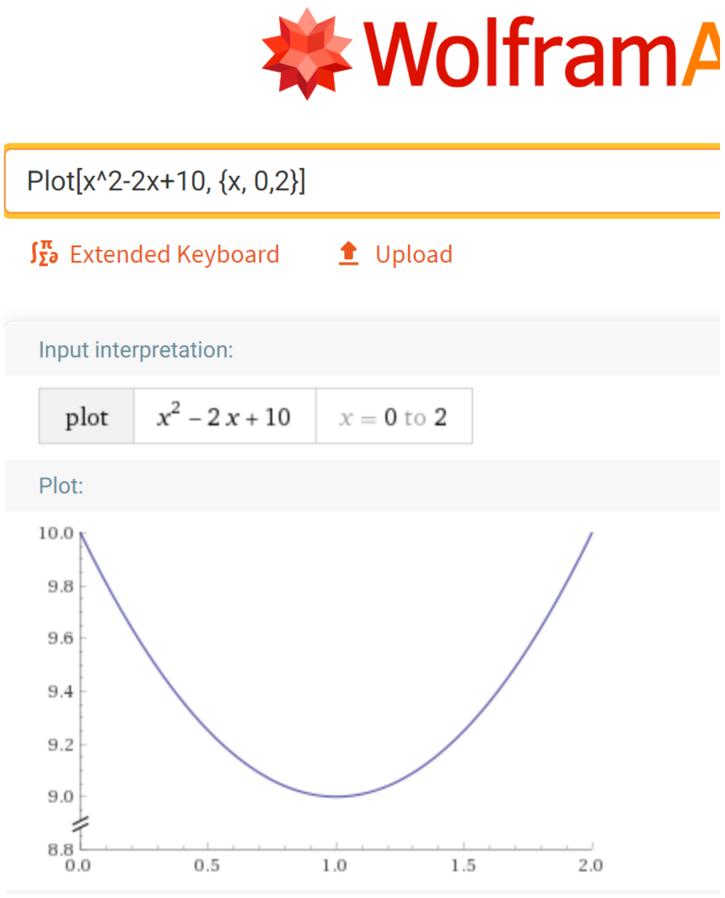
- The characteristic equation is

$$\lambda^2 - 2\lambda + 10 = 0$$

- What are the eigenvalues?

Complex Eigenvalues

- Note that there is no real solution to
  $\lambda^2 - 2\lambda + 10 = 0$
- To see this, visit https://www.wolframalpha.com/ and execute
  $$\mathrm{Plot}[x\text{^}2 - 2x + 10, \{x, 0{,}2\}]$$
- As you can see, the curve has no intersection with the horizontal axis.



Complex Eigenvalues

- Consider the characteristic equation:

$$\lambda^2 - 2\lambda + 10 = 0$$

- What are the eigenvalues?
- Remember the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Thus, the solutions to $\lambda^2 - 2\lambda + 10 = 0$ must be

$$\lambda = \frac{2 \pm \sqrt{4 - 4\times 10}}{2} = 1 \pm \sqrt{-9}$$

Complex Eigenvalues

- Never seen $1 + \sqrt{-9}$ before?
- This is an example of **complex numbers**.
- Let i satisfy $i^2 = -1$.
- Then i is called an **imaginary number**.
- Let a and b real numbers. Then $a + bi$ is a **complex number** and $a - bi$ its **complex conjugate**.
- In our example, the eigenvalues are $1 \pm 3i$.
- Can we take the absolute value of a complex number such that $|1 + 3i|$?

Complex Eigenvalues

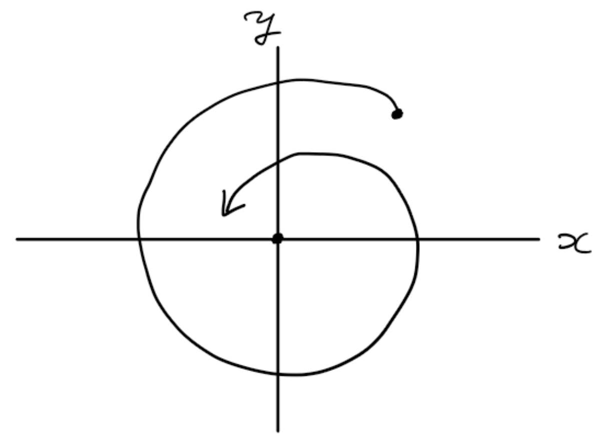
- To consider the length of a complex number, we introduce the concept of modulus.
- Consider a complex number $a + bi$. Its **norm** or **modulus** is given by

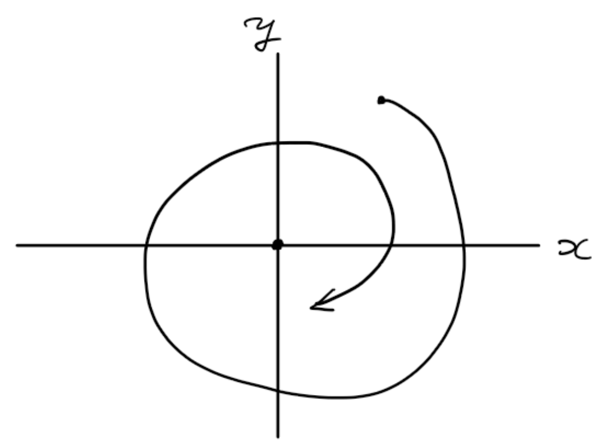
$$r = \sqrt{a^2 + b^2}$$

- When $b = 0$, we have a real number a and its modulus is $r = \sqrt{a^2} = |a|$.
- Thus, the concept of modulus includes the absolute value of a real number as a special case.

Complex Eigenvalues

- $r = \sqrt{a^2 + b^2} < 1$
  $\Longrightarrow$ **Spiral sink**.
- Consider
  $$A = \begin{pmatrix} 0.1 & 0.1 \\ -2 & 0.1 \end{pmatrix}$$
- The eigenvalues are
  $$\lambda = 0.1 \pm 0.45i$$
- In Octave or Matlab, execute "abs(eig(A))" to obtain $r = 0.458 < 1$.



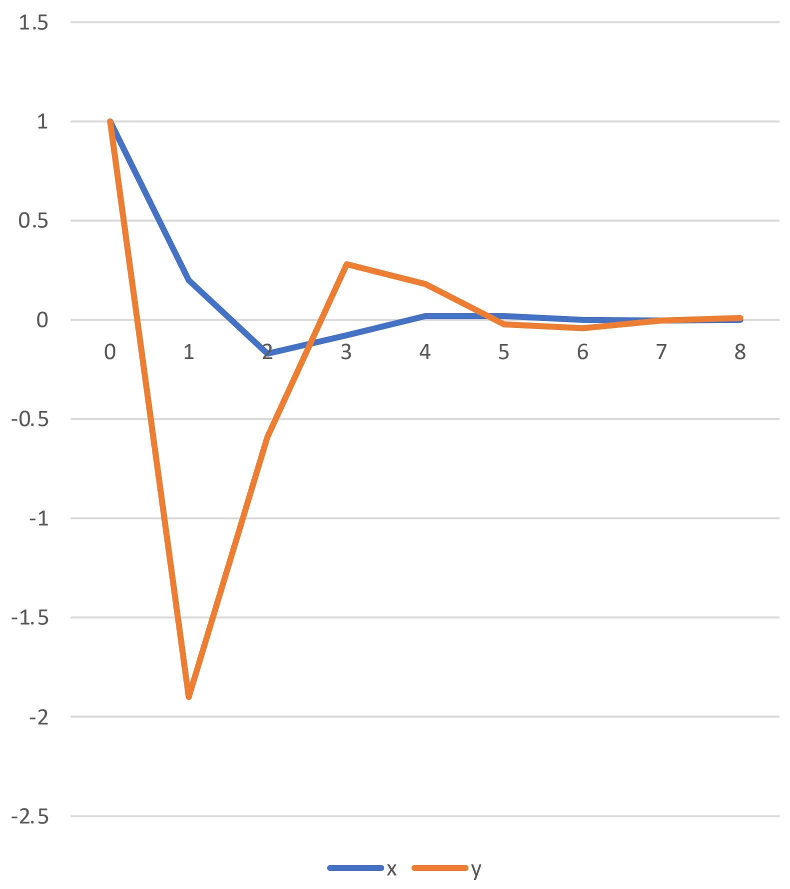


Complex Eigenvalues

- Consider

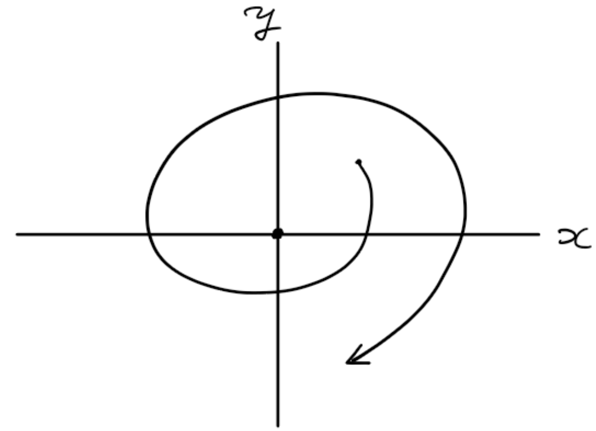
$$A = \begin{pmatrix} 0.1 & 0.1 \\ -2 & 0.1 \end{pmatrix}$$

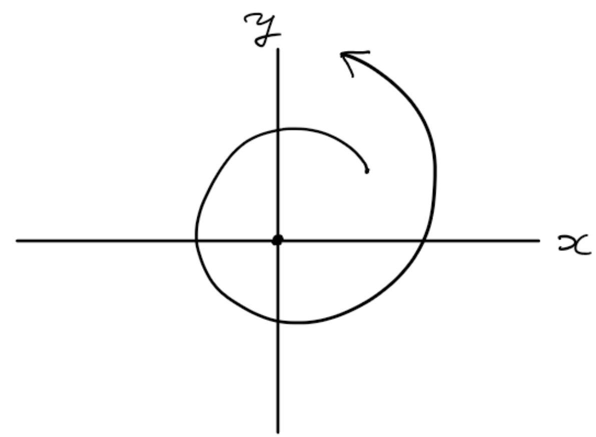
- $r = 0.458 < 1.$
- Dynamical systems with complex eigenvalues generate rich time series patterns.



Complex Eigenvalues

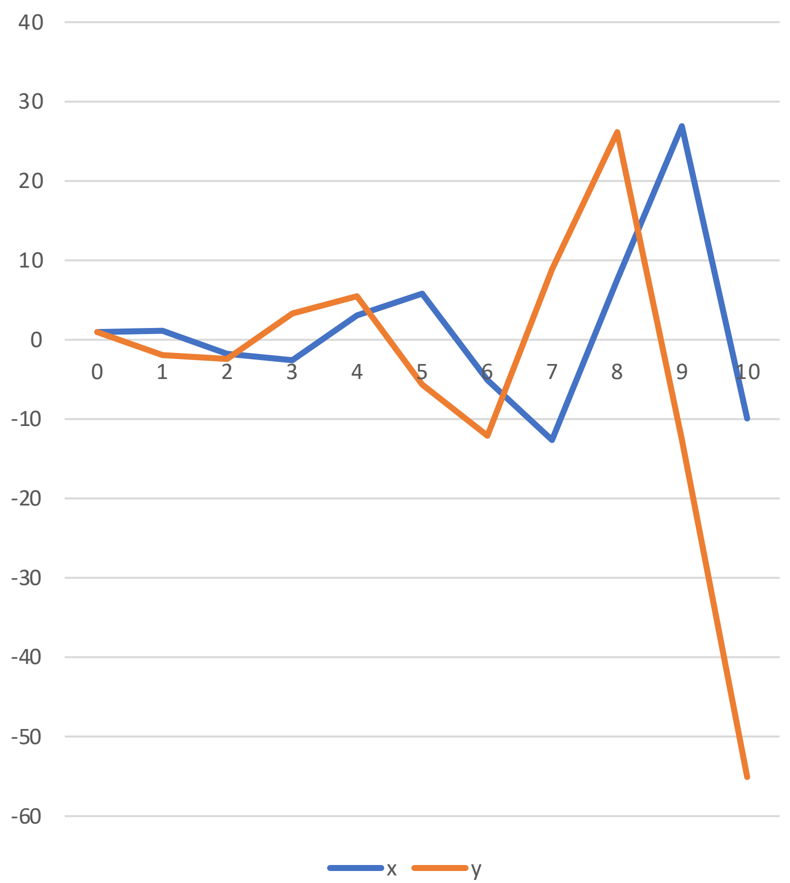
- $r = \sqrt{a^2 + b^2} > 1$
  $\Longrightarrow$ **Spiral source**.
- Consider
  $$A = \begin{pmatrix} 0.1 & 1 \\ -2 & 0.1 \end{pmatrix}$$
- The eigenvalues are
  $$\lambda = 0.1 \pm 1.41i$$
- In Octave or Matlab, execute "abs(eig(A))" to obtain $r = 1.418 > 1$.





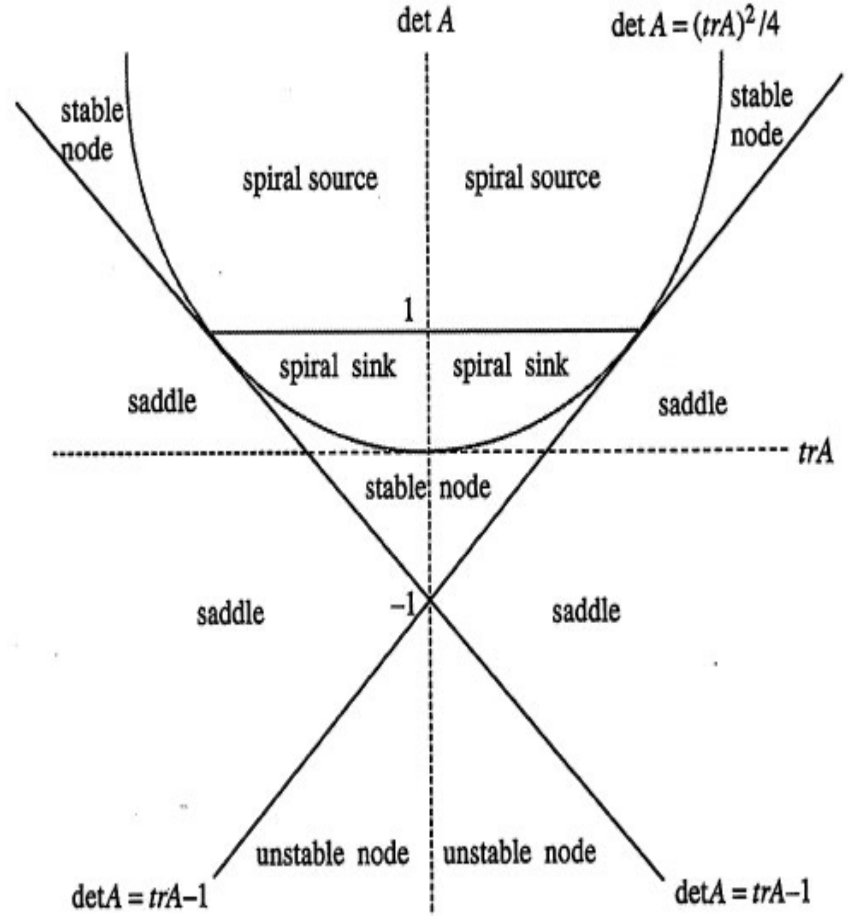
Complex Eigenvalues

- Consider
$$A = \begin{pmatrix} 0.1 & 1 \\ -2 & 0.1 \end{pmatrix}$$
- $r = 1.418 > 1.$



Summary

- Consider a two-dimensional linear system: $z_{t+1} = Az_t$
- Characteristic equation: $\lambda^2 - \mathrm{tr}A\lambda + \det A = 0$



Oded Galor, *Discrete Dynamical Systems*, Springer, 2010.

Higher-Dimensional Systems

- Consider a higher-dimensional linear system:

$$\begin{pmatrix} x_{t+1}^1 \\ \vdots \\ x_{t+1}^n \end{pmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} x_t^1 \\ \vdots \\ x_t^n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- More compactly, $x_{t+1} = Ax_t + b$
- Steady state is $x = -(A - I)^{-1}b$
- The general solutions is

$$x_t = x + \sum_{i=1}^{n} c_i \lambda_i^t P_i$$

Higher-Dimensional Systems

- **Theorem:** Suppose all eigenvalues of A have moduli different from 1. Then the unique steady state of the system, x, is globally (asymptotically) stable if and only if all the eigenvalues of A have moduli strictly smaller than 1, and unstable if at least one eigenvalue has modulus strictly larger than 1.
- Definition: A steady state x is **globally (asymptotically) stable** if $\lim_{t\to\infty} x_t = x$ for any x_0.

Further Readings

- Simon and Blume, *Mathematics for Economists*, Norton, 1994.
  - Chapter 23 “Eigenvalues and Eigenvectors” is made available for download from TACT.
- Sydsaeter, Hammond, Seierstad, and Strom, *Further Mathematics for Economic Analysis*, 2nd edition, Prentice Hall, 2008.