Lecture 3

Chapter 1: Difference Equations

Part III: Linear Systems

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Linear Systems

- Given a set of parameters a, b, c, d, we consider
$$\begin{cases} x_{t+1} = ax_t + by_t \\ y_{t+1} = cx_t + dy_t \end{cases}$$
- This is a **first-order linear system of difference equations**.
- In matrix form,
$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$
- Because the matrix is 2 by 2, this is a two-dimensional system.

Linear Systems

- Define a column vector z_t such that

$$z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- Define a 2×2 square matrix such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Then, our system of DE can be written compactly as

$$z_{t+1} = Az_t$$

- This system is homogeneous as it has no constant term.

Linear Systems

- Now consider a **second-order** equation:

$$x_{t+1} = ax_t + bx_{t-1}$$

- Defining $y_t = x_{t-1}$ allows us to rewrite the equation as

$$x_{t+1} = ax_t + by_t$$
$$y_{t+1} = x_t$$

- In matrix form,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

- Higher-order equations can always reduce to <u>higher-dimensional first-order systems</u>.
  - We no longer need to study higher-order equations separately.

Linear Systems

- Today we shall focus on $z_{t+1} = Az_t$, or
$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$
- I hope you are now convinced that this system is sufficiently general.
- How can we solve it analytically?
- Too hard? Then how about this?
$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Linear Systems

- When $b = c = 0$, the system reduces to

$$\begin{cases} x_{t+1} = a x_t \\ y_{t+1} = d y_t \end{cases}$$

- We can solve them separately as

$$\begin{cases} x_t = a^t x_0 \\ y_t = d^t y_0 \end{cases}$$

- In other words, for any diagonal matrix A, we can solve the system immediately.

Linear Systems

- Our solution strategy is to
  - transform $z_{t+1} = Az_t$ into another system with a diagonal matrix,
  - solve the transformed DE, and
  - recover the original solution from the transformed solution.
- As an example, in what follows we shall consider

$$A = \begin{pmatrix} 1 & 1.5 \\ 0.5 & 0 \end{pmatrix}$$

Linear Systems

- There are 4 steps to solve the system of DE:
  1. Find the eigenvalues of A and eigenvectors.
  2. Diagonalize A and obtain the transformed system
  3. Solve the transformed system
  4. Recover the original solution form the transformed solution.
- In what follows, we need to work with matrices.
  - I made Chapter 23 of Simon and Blume available at TACT for your convenience.

Step 1: Eigenvalues and Eigenvectors of A

- Definition: Consider a square matrix A. An **eigenvalue** of A is a number λ which converts A into a **singular matrix** by subtracting λ from each of the diagonal entries of A.
- Thus, λ is an eigenvalue of A if and only if $A - \lambda I$ is **singular**, where I is the **identity matrix**.
  - A square matrix is **singular** if and only if its **determinant** is zero.
- In other words, λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Step 1: Eigenvalues and Eigenvectors of A

- Let us calculate $\det(A-\lambda I)$.

$$\det(A-\lambda I) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$
$$= (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - \underbrace{(a+d)}_{\mathrm{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

- Thus, λ is found by solving

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

Step 1: Eigenvalues and Eigenvectors of A

- $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ is called the **characteristic equation** of A.
- Because this equation is quadratic, there are two solutions, λ_1 and λ_2, say.
- The solutions to the characteristic equation are the **eigenvalues** of A.
  - They are also known as the **characteristic roots**.

Step 1: Eigenvalues and Eigenvectors of A

- Consider our example:

$$A = \begin{pmatrix} 1 & 1.5 \\ 0.5 & 0 \end{pmatrix}$$

- The corresponding characteristic equation is

$$\lambda^2 - (1+0)\lambda + (0 - 1.5 \times 0.5) = 0$$
$$\Leftrightarrow (\lambda - 1.5)(\lambda + 0.5) = 0$$
$$\Longrightarrow \lambda_1 = -0.5, \lambda_2 = 1.5$$

  - It is OK to label $\lambda_1 = 1.5, \lambda_2 = -0.5$, but I chose λ_1 to be the smaller one in absolute value.

Step 1: Eigenvalues and Eigenvectors of A

- Consider a nonzero vector P.
- $(A - \lambda I)$ is singular if and only if the system of equations $(A - \lambda I)P = 0$ has a nonzero solution.
  - Such a nonzero vector P is called an **eigenvector** of A.
- A two-dimensional system has two eigenvalues, and we can find an eigenvector for each eigenvalue.
- Let $P_1 = \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$ correspond to λ_1.
- Let $P_2 = \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}$ correspond to λ_2.

Step 1: Eigenvalues and Eigenvectors of A

- Consider $P_1 = \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$ for $\lambda_1 = -0.5$. Then,
  $$(A - \lambda_1 I)P_1 = 0 \Leftrightarrow \begin{pmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = 0$$
- In our example, we have
  $$\begin{pmatrix} 1.5 & 1.5 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = 0 \Leftrightarrow \begin{cases} 1.5e_{11} + 1.5e_{12} = 0 \\ 0.5e_{11} + 0.5e_{12} = 0 \end{cases}$$
- What is the solution?

Step 1: Eigenvalues and Eigenvectors of A

- Consider

$$\begin{cases} 1.5e_{11} + 1.5e_{12} = 0 \\ 0.5e_{11} + 0.5e_{12} = 0 \end{cases}$$

- Notice that one equation is a linear transformation of the other (i.e., one equation is redundant).
- Consider $e_{11} + e_{12} = 0$.
  - One equation in two unknowns.
  - There is an infinity of solutions.

Step 1: Eigenvalues and Eigenvectors of A

- Consider $e_{11} + e_{12} = 0$.
- Any combination of e_{11} and e_{12} satisfying $e_{11} + e_{12} = 0$ is a solution.
- Examples are:

$$P_1 = \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \dots$$

- Before going to the next page, pick up your paper and pencil to calculate examples of P_2.

Step 1: Eigenvalues and Eigenvectors of A

- Consider $P_2 = \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}$ for $\lambda_2 = 1.5$. Then,

$$(A - \lambda_2 I)P_2 = 0$$

$$\Leftrightarrow \begin{pmatrix} 1-1.5 & 1.5 \\ 0.5 & 0-1.5 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} -0.5 & 1.5 \\ 0.5 & -1.5 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{cases} -0.5e_{21} + 1.5e_{22} = 0 \\ 0.5e_{21} - 1.5e_{22} = 0 \end{cases}$$

- Thus, examples of P_2 are

$$P_2 = \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \dots$$

Step 1: Eigenvalues and Eigenvectors of A

- Open your Octave or Matlab.
- In Octave or Matlab, we can calculate the eigenvalues and eigenvectors within a second.
- In the command window, execute the following:

```
A=[1,1.5;0.5,0]
```

- It defines matrix A. Then, execute the following:

```
eig(A)
```

- Finally, execute the following:

```
[V,D] = eig(A)
```

Matlab/Octave: Eigenvalues and Eigenvectors of A

- How can Octave (or Matlab) choose one particular eigenvector?
- From an infinity of vector satisfying $e_{11} + e_{12} = 0$, the one satisfying $e_{11}^2 + e_{12}^2 = 1$ is chosen.
- This is called the **unit eigenvector**.
  - Meaning, the length of the vector is 1.

```
>> A=[1,1.5;0.5,0]
A =

   1.00000   1.50000
   0.50000   0.00000

>> eig(A)
ans =

   1.50000
  -0.50000

>> [V,D]=eig(A)
V =

   0.94868  -0.70711
   0.31623   0.70711


D =

Diagonal Matrix

   1.50000         0
         0  -0.50000
```

Python: Eigenvalues and Eigenvectors of A

- In Python, we need a longer code to do the same.

```
import numpy as np

a = np.array([[1, 1.5],
              [0.5, 0]])

a_eig = np.linalg.eig(a)
val, vec = np.linalg.eig(a)
print(val)
print(vec)
```

```
[ 1.5 -0.5]
[[ 0.9486833  -0.70710678]
 [ 0.31622777  0.70710678]]
```

Step 2: Diagonalizing A

- Define a new matrix E as
$$E = (P_1, P_2) = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$$
- First, let us check if E is **invertible**.
- $\Leftrightarrow E$ is nonsingular, which is satisfied because
$$\det \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} = 4 \neq 0$$
- Pick up your paper and pencil to find the **inverse** of E.

Step 2: Diagonalizing A

- Remember that the **inverse** of A is

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- Apply this rule to obtain

$$E = \begin{pmatrix} 0.25 & -0.75 \\ 0.25 & 0.25 \end{pmatrix}$$

- You can verify this using Octave by `inv(E)`, or using Python by `np.linalg.inv(A)`.
- Use your paper and pencil to calculate $E^{-1}AE$.

Step 2: Diagonalizing A

- Consider $E^{-1}AE$. After several lines of calculation (or, one line in Octave or Matlab), you should be able to verify that
$$E^{-1}AE = \begin{pmatrix} -0.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$
- Surprisingly, this magical manipulation gives us
$$E^{-1}AE = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
- For convenience, we shall define a matrix
$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Step 2: Diagonalizing A

- **Pre-multiply** both sides of $z_{t+1} = Az_t$ by E^{-1}:

$$\begin{aligned} E^{-1}z_{t+1} &= E^{-1}Az_t \\ &= E^{-1}A(EE^{-1})z_t \\ &= (E^{-1}AE)E^{-1}z_t \\ &= \Lambda E^{-1}z_t \end{aligned}$$

- Define

$$\hat{z}_t = E^{-1}z_t$$

- Then, we obtain

$$\hat{z}_{t+1} = \Lambda\hat{z}_t$$

- This is the transformed system.

Step 3: Solving $\hat{z}_{t+1} = \Lambda \hat{z}_t$

- Let the elements of $\hat{z}_t$ be

$$\hat{z}_t = \begin{pmatrix} \hat{x}_t \\ \hat{y}_t \end{pmatrix}$$

- The transformed system $\hat{z}_{t+1} = \Lambda \hat{z}_t$ reduces to

$$\begin{cases} \hat{x}_{t+1} = \lambda_1 \hat{x}_t \\ \hat{y}_{t+1} = \lambda_2 \hat{y}_t \end{cases}$$

- Then, we can solve them <u>separately</u> as

$$\begin{cases} \hat{x}_t = \lambda_1^t \hat{x}_0 \\ \hat{y}_t = \lambda_2^t \hat{y}_0 \end{cases}$$

Step 4: Recovering the Original Solution

- Remember how $\hat{z}_t$ is defined.

$$\hat{z}_t = E^{-1} z_t$$

- Pre-multiply both sides by E to obtain

$$E\hat{z}_t = EE^{-1} z_t = z_t$$

- It tells us how to transform $\hat{z}_t$ back into the original solution z_t. Thus, the solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^t \hat{x}_0 \\ \lambda_2^t \hat{y}_0 \end{pmatrix}$$

Step 4: Recovering the Original Solution

- We can rewrite the solution as

$$\begin{cases} x_t = \underbrace{c_1 \, e_{11}}_{\tilde{x}_0} \lambda_1^t + \underbrace{c_2 \, e_{21}}_{\tilde{y}_0} \lambda_2^t \\ y_t = \underbrace{c_1 \, e_{12}}_{\tilde{x}_0} \lambda_1^t + \underbrace{c_2 \, e_{22}}_{\tilde{y}_0} \lambda_2^t \end{cases}$$

- Or, more compactly,

$$z_t = c_1 P_1 \lambda_1^t + c_2 P_2 \lambda_2^t$$

- This is referred to as the **general solution**.
  - Because we are not specifying the initial condition.

Step 4: Recovering the Original Solution

- Consider the solution:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^t \hat{x}_0 \\ \lambda_2^t \hat{y}_0 \end{pmatrix}$$

- Notice

$$\begin{pmatrix} \lambda_1^t \hat{x}_0 \\ \lambda_2^t \hat{y}_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} \hat{x}_0 \\ \hat{y}_0 \end{pmatrix}$$

- Thus, the solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = E \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} E^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Non-Homogeneous Systems

- Consider $z_{t+1} = Az_t + B$, where B is a column vector with parameters.
- This system is not homogeneous.
- Let z denote the steady state vector. Then,
$$z = Az + B$$
- To find the steady state, manipulate the equation:
$$\begin{aligned} z = Az + B &\Leftrightarrow -B = (A - I)z \\ &\Leftrightarrow -(A - I)^{-1}B = z \end{aligned}$$

Non-Homogeneous Systems

- Subtract z from both sides of $z_{t+1} = Az_t + B$:
$$z_{t+1} - z = Az_t + B - z$$
$$= Az_t + B - Az - B$$
$$= A(z_t - z)$$
- Define $Z_t = z_t - z$ to obtain
$$Z_{t+t} = AZ_t$$
- We know how to solve it: $Z_t = c_1 P_1 \lambda_1^t + c_2 P_2 \lambda_2^t$.
- Thus, the solution to the non-homogeneous system is given by
$$z_t = z + Z_t = z + c_1 P_1 \lambda_1^t + c_2 P_2 \lambda_2^t$$

Higher Dimensional Systems

- Consider the n-dimensional non-homogeneous system:

$$\begin{pmatrix} x_{t+1}^1 \\ \vdots \\ x_{t+1}^n \end{pmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} x_t^1 \\ \vdots \\ x_t^n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- More compactly, $x_{t+1} = Ax_t + b$.
- The steady state is $x = -(A - I)^{-1}b$
- The general solutions is given by

$$x_t = x + \sum_{i=1}^{n} c_i \lambda_i^t P_i$$

Further Readings

- Simon and Blume, *Mathematics for Economists*, Norton, 1994.
  - Chapter 23 “Eigenvalues and Eigenvectors” is made available for download from TACT.
- Sydsaeter, Hammond, Seierstad, and Strom, *Further Mathematics for Economic Analysis*, 2nd edition, Prentice Hall, 2008.