Lecture 2

Chapter 1: Difference Equations

Part II: Nonlinear Equations and Linearization

4/20, 2023

Introductory Example

- Consider

$$x_{t+1} = a(x_t^2 - 3x_t + 6)$$

- This is a first-order **non-linear** difference equation.

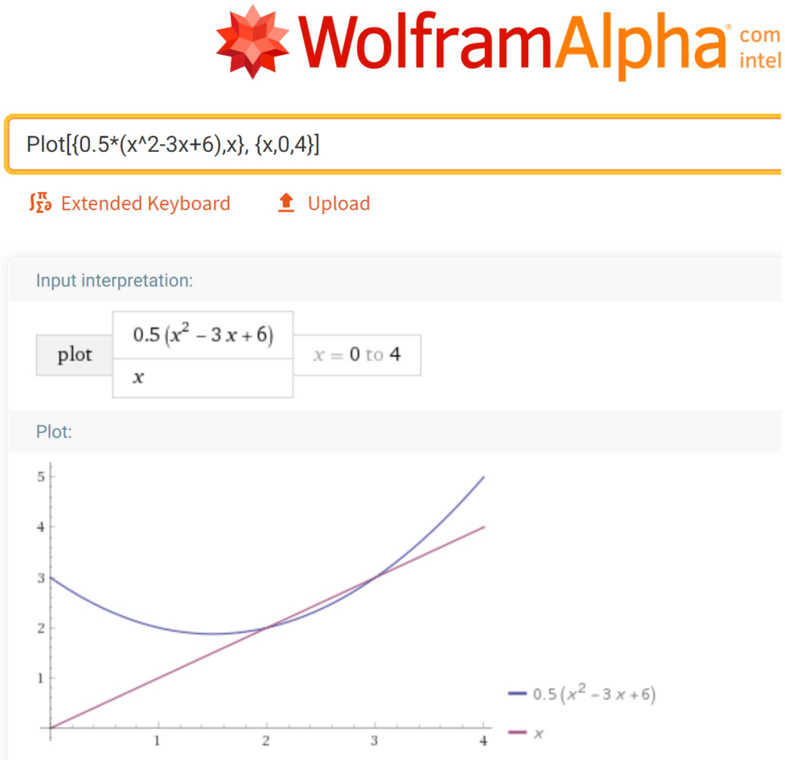
1. Suppose $a = 0.5$. Find the steady state by drawing a diagram.
2. Suppose $a = 1$. Find the steady state by drawing a diagram.

Introductory Example

- First, consider the case with $a = 0.5$. Then,
$$x_{t+1} = 0.5(x_t^2 - 3x_t + 6)$$
- Let x_{ss} denote the steady state of the equation.
- Then, the steady state must satisfy
$$x_{ss} = 0.5(x_{ss}^2 - 3x_{ss} + 6)$$
- In diagram, the left-hand side is the 45-degree line.
- We want to draw the right-hand side correctly.
- Today, let me cheat by using WalframAlpha at https://www.wolframalpha.com/.

Introductory Example

- At WolframAlpha, type and execute the following:
  $\text{Plot}[\{0.5*(x\text{^}2-3x+6), x\}, \{x, 0{,}4\}]$
  - The language is the same as Mathematica.



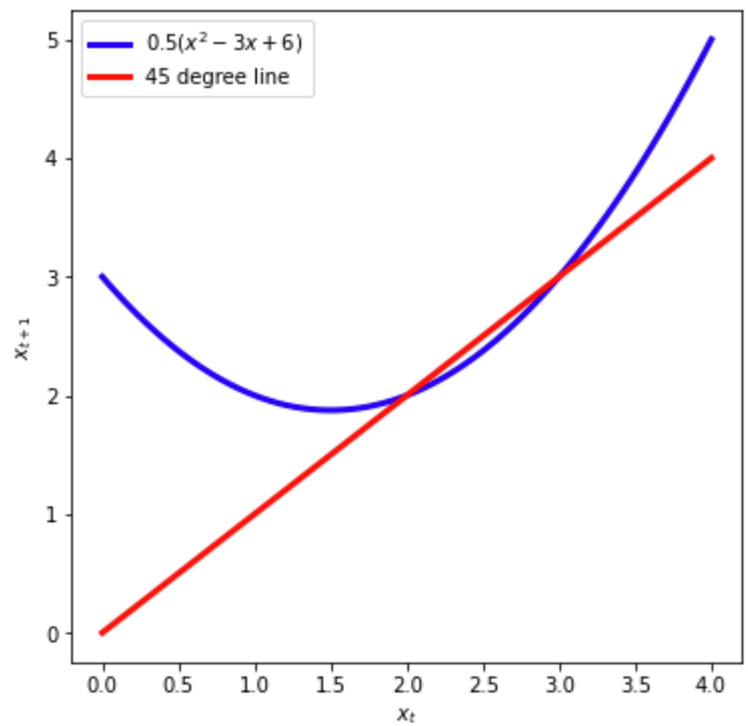
- As you can see, there are <u>two</u> intersections.
- Thus, this DE has <u>two</u> <u>steady states</u>.

In Python

The following code is available (as a text file) at TACT

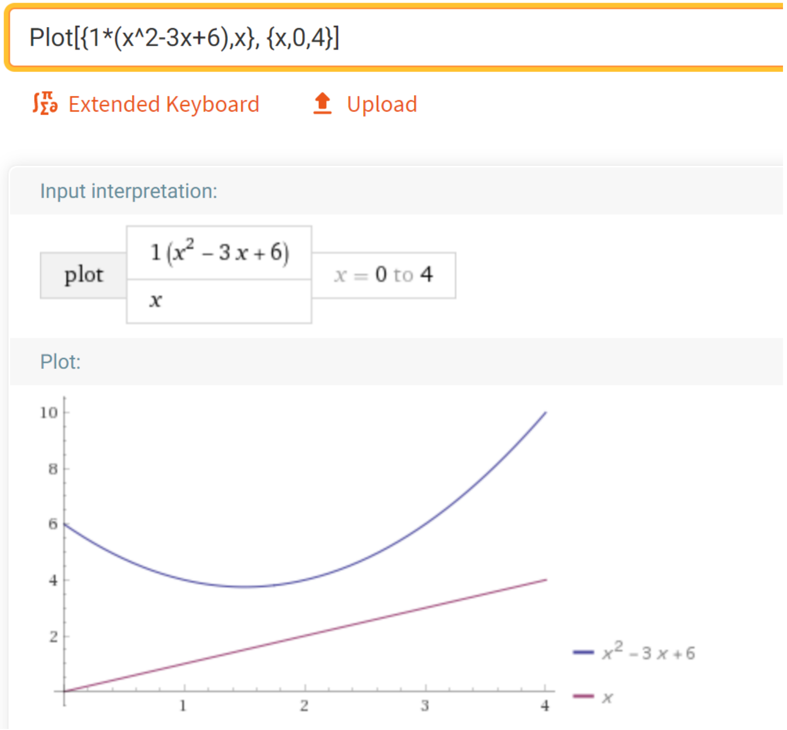
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['figure.figsize'] = (6,6)

grid = np.linspace(0, 4, 100)
fig, ax = plt.subplots()

eq = 0.5*(grid**2 -3*grid +6)
dl = grid

ax.plot(grid, eq, 'b-', lw=3, label='$0.5(x^2 -3x+6)$')
ax.plot(grid, dl, 'r-', lw=3, label='45 degree line')

ax.set_xlabel('$x_t$')
ax.set_ylabel('$x_{t+1}$')
ax.legend()

plt.show()
```

I learned Python codes at https://python-programming.quantecon.org/intro.html



Introductory Example

- At WolframAlpha, type and execute the following:
  Plot[{1 ∗ (x^2 − 3x + 6), x}, {x, 0,4}]
- As you can see, there is <u>no</u> intersection.
- A steady state <u>does not exist</u> for this DE.



Observations

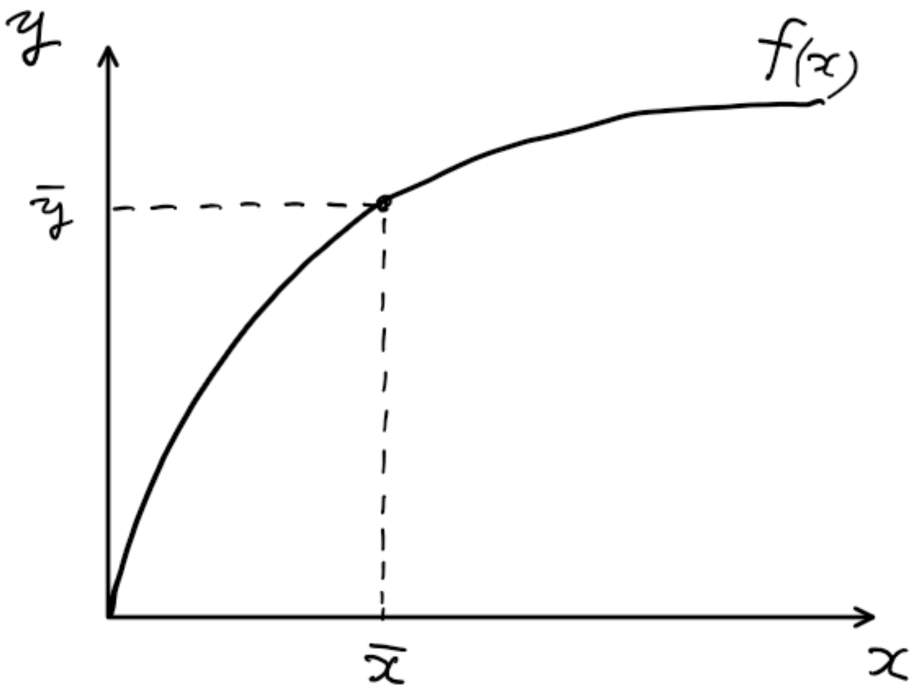
- Consider $x_{t+1} = f(x_t)$, where f is nonlinear.
- Observations:

1. A steady state does **not necessarily exist**.
2. There can be **more than one** steady state.

- Whenever you encounter a nonlinear DE, the very first thing to do is counting the number of steady states.

Linearization

- Consider $y = f(x)$, where f is nonlinear.
- $(\bar{x}, \bar{y})$ is a particular point on the diagram.

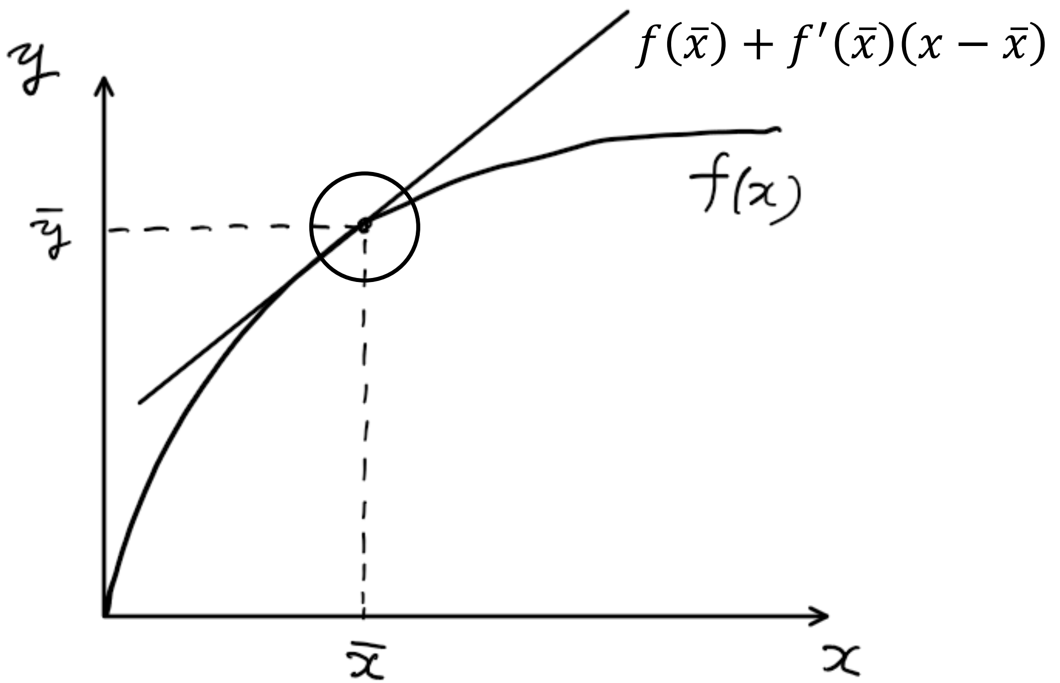


Linearization

- The derivative of $f(x)$ at $x = \bar{x}$ is defined as
$$f'(\bar{x}) = \lim_{h} \frac{f(\bar{x}+h) - f(\bar{x})}{h}$$
- Thus, for small h,
$$f'(\bar{x})h \approx f(\bar{x}+h) - f(\bar{x})$$
- Let $h = x - \bar{x}$. Then we can rewrite the above as
$$f'(\bar{x})(x-\bar{x}) \approx f(x) - f(\bar{x})$$
- Thus, for x close enough to $\bar{x}$, we can linearly approximate $f(x)$ at $\bar{x}$ as
$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x})$$

Linearization

- Planet earth is a ball. But, to us, it is flat enough.
- Likewise, any differentiable function is locally linear.



Example

- Consider $f(x) = \log x$. Linearize it at $x = \bar{x}$.
- For x close to $\bar{x}$, we have

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$$

- Because $(\log x)' = 1/x$, we rewrite the above as

$$\log x = \log \bar{x} + \frac{x - \bar{x}}{\bar{x}}$$

- This implies

$$\log x - \log \bar{x} = \frac{x - \bar{x}}{\bar{x}}$$

- Thus, <u>a log difference means a percentage change</u>.

Solving Nonlinear DE

- Now consider nonlinear DE: $x_{t+1} = f(x_t)$
- Let us assume (at least) one steady state x_{ss} exists.
- Evidently, x_{ss} satisfies $x_{ss} = f(x_{ss})$.
- Linearize $f(x_t)$ around the steady state to obtain
$$f(x_t) = f(x_{ss}) + f'(x_{ss})(x_t - x_{ss})$$
- Thus, our linearized DE is
$$x_{t+1} = f(x_{ss}) + f'(x_{ss})(x_t - x_{ss})$$
- Remember $x_{ss} = f(x_{ss})$. Thus,
$$x_{t+1} = x_{ss} + f'(x_{ss})(x_t - x_{ss})$$

Solving Nonlinear DE

- Thus,

$$x_{t+1} - x_{ss} = f'(x_{ss})(x_t - x_{ss})$$

- Let

$$\hat{x}_t = x_t - x_{ss} \; or \; \hat{x}_t = \frac{x_t - x_{ss}}{x_{ss}}$$

- Then, we obtain

$$\hat{x}_{t+1} = f'(x_{ss})\hat{x}_t$$

- Because x_{ss} is a number, $f'(x_{ss})$ is a number, too.
- Thus, we let

$$a = f'(x_{ss})$$

Solving Nonlinear DE

- We finally obtain

$$\hat{x}_{t+1} = a\hat{x}_t$$

- You should be able to tell the solution to this DE within a second.

Solving Nonlinear DE

- The solution to $\hat{x}_{t+1} = a\hat{x}_t$ is
$$\hat{x}_t = a^t \hat{x}_0$$
- From our previous class, we know the stability conditions.
- We say that the steady state x_{ss} is **locally stable** if and only if $|a| < 1$.
- In other words, for $x_{t+1} = f(x_t)$, steady state x_{ss} is locally stable if and only if
$$|f'(x_{ss})| < 1$$

Solving Nonlinear DE

- We can linearize $x_{t+1} = f(x_t)$ at x_{ss} by $\hat{x}_{t+1} = f'(x_{ss})\hat{x}_t$. But our method takes many lines.
- We need more speed.
- Total differentiation can help us obtain the same expression within a second.
  - If you have no idea about total differentiation, go back to your math textbook immediately.

Solving Nonlinear DE

- Consider $x_{t+1} = f(x_t)$ with $x_{ss} = f(x_{ss})$.
- Totally differentiate our DE to obtain
$$dx_{t+1} = f'(x_t)dx_t$$
- Evaluate each coefficient at its steady state level:
$$dx_{t+1} = f'(x_{ss})dx_t$$
- We are done!
- Note that dx_t is a small change in x_t from its point of evaluation, x_{ss}. Thus,
$$dx_t = x_t - x_{ss}$$

Solving Nonlinear DE

- Note that $dx_t = x_t - x_{ss}$ measures a distance.
- In many applications, it is much more convenient to focus on a percentage deviation from x_{ss}:
$$\frac{x_t - x_{ss}}{x_{ss}} = \hat{x}_t$$
- Totally differentiate $x_{t+1} = f(x_t)$, evaluate the coefficient at x_{ss}, and divide both sides by x_{ss} to obtain
$$\hat{x}_{t+1} = f'(x_{ss})\hat{x}_t$$
- This method takes one second.

Solving Nonlinear DE

- Our result on page 11 implies that
$$\hat{x}_t = \frac{x_t - x_{ss}}{x_{ss}} = \log x_t - \log \bar{x}$$
- In this sense, $\hat{x}_{t+1} = f'(x_{ss})\hat{x}_t$ is sometimes called a **log-linearized equation**. And the procedure is called **log-linearization**.

Example

- Consider $x_{t+1} = 0.5(x_t^2 - 3x_t + 6)$.

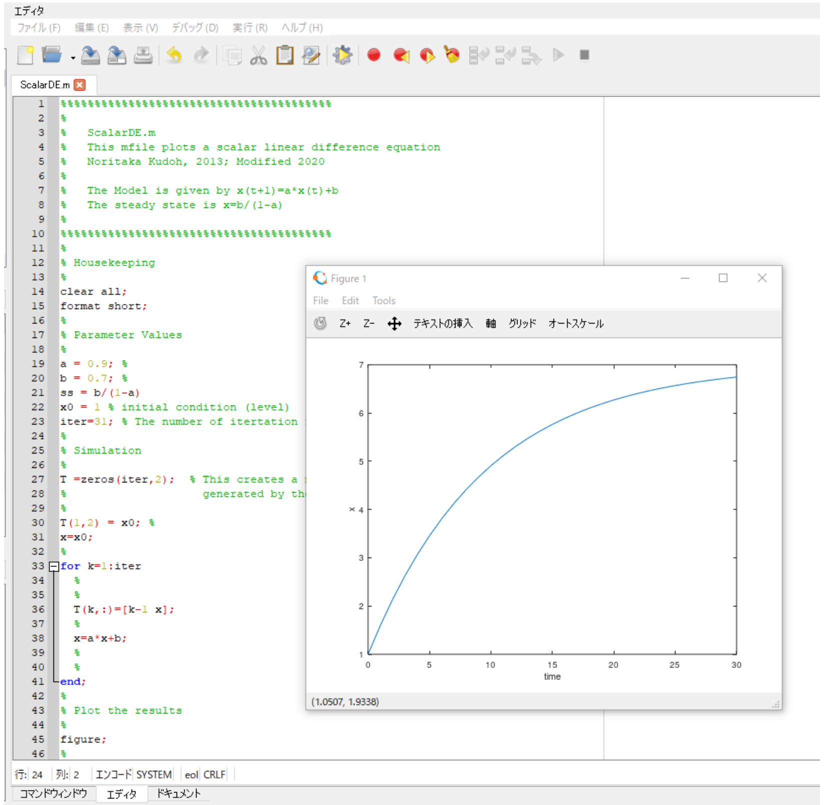
a) Find all steady states.

b) Log-linearize this equation around **each** of the steady states.

c) For each log-linearized equation, study its stability property.

Running a Matlab Code

- Obtain “ScalarDE.m” from TACT.
- Prepare a new folder and save the file.
- In Octave or Matlab, open the folder and execute the file.
- Read the code to study the language.



Running a Python Code

- Obtain "PythonCode_ScalarDE.txt" from TACT.
- Open the file in any text editor (such as memo pad in windows pc).
- Copy and paste the code in Jupyter Notebook (or Colab).
- Read the code to study the language.

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

ts_length = 31
x_values = []    # empty list
a = 0.9
b = 1
x0 = 1
xt = x0

for i in range(ts_length):
    xt = a*xt + b
    x_values.append(xt)

plt.plot(x_values)
plt.show()
```