Advanced Macroeconomics I

「上級所得理論Ｉ」 for graduate program
「Advanced Income Theory I」 for G30 and NUPACE

Lecture 1

Introduction to advanced macroeconomics

4/13, 2023

About Me

- Noritaka Kudoh, Ph.D.
- Professor of Economics at Nagoya University
- Macroeconomist
- Website: https://sites.google.com/site/noritakakudoh/home

Goals

- Today, we have two goals.
- In the Course Description part, we understand the key differences between the undergraduate macroeconomics and the graduate-level macroeconomics.
- In the Linear Scalar Difference Equations part, we study a new mathematical tool for macroeconomic analysis.

Course Description

Where We Stand

- Since the 1980s, macroeconomics has made a remarkable progress especially in terms of its methodology.
- **The gap between undergraduate macro and graduate macro has become enormous**.
- As a result, it is **entirely hopeless** to understand any of the research papers written in the last 40 years without a strong methodological background.
- What do latest academic papers look like?

What do academic papers look like?

- Take a brief look at my papers published in 2019 and in 2023:
  - https://tact.ac.thers.ac.jp/x/x15Oeu
  - https://tact.ac.thers.ac.jp/x/DDmji7
- There are two major departures from undergrad macro:
  1. Dynamics: Variables change their values over time.
  2. Optimization: All decisions are derived, not assumed.

Undergraduate Macro Model

- The model is static (= concept of time is absent).
- National income identity (closed economy):
$$Y = C + I + G$$
- Consumption function:
$$C = aY + b$$
- The equilibrium (= solution) of the model is obtained by solving a system of equation.

Difference Equations

- National income identity: $Y_t = C_t + I + G$
- Consider: $C_t = aY_{t-1} + b$
- The equilibrium GDP satisfies
$$Y_t = aY_{t-1} + b + I + G$$
- There are Y_t and Y_{t-1} in one equation.
- This is an example of **difference equation**.

Undergraduate Macro Model

- National income identity: $Y = C + I + G$
- Consumption function: $C = aY + b$
- Consumption function model is ad-hoc (= the household's action is arbitrarily chosen by us, not by individuals in the model).
- What does **micro**economics tell us?
    - Households make decisions to maximize utility.

Dynamic Optimization

- Instead of we choosing consumption, we introduce an individual who chooses his/her consumption.
- A typical dynamic optimization looks like:

$$\max_{\{c_t\}_{t=0}^{\infty},\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = f(k_t) + (1+\delta)k_t - c_t$$

for $t = 0 \ldots \infty$.

Goal for this Semester

- We focus on particularly important methodological skills that are often used in modern macroeconomic research:
  1. **difference equations** for describing variables that evolve over time, and
  2. **dynamic optimization** for describing the optimal allocation of resources over time
- After this semester, you will be able to read professional articles and be ready to catch up with the research frontier. This is our goal.

Long Run Goals

- Requirement for MA thesis
  - Fully understand and replicate some results published in a leading professional journal.
  - (Optional) Modify the analysis (by a new data set or by a new parameter set) to add some originality.
- Requirement for Ph.D. dissertation
  - Introduce innovative assumptions or methods to the existing analysis to find something **new**.
  - Complete 3 papers, one of which must be published.

Target

- This course is designed for first-year graduate students.
  - But, all highly-motivated students are welcome.
  - However, please fully understand that this course is demanding (= many time-consuming assignments).
- To help you acquire skills, this course is <u>intentionally made painful for you</u>.

Prerequisites

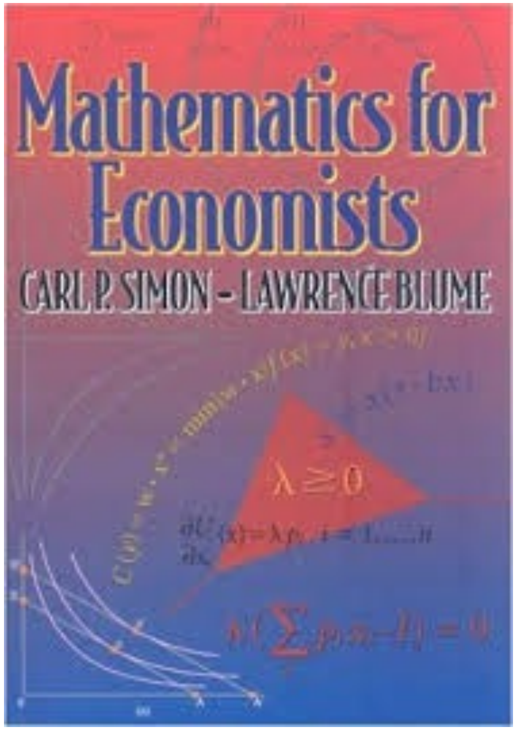
- I will assume that you are familiar with:
  - Constrained maximization problems using the Lagrangian and Karush-Kuhn-Tucker (formally known as Kuhn-Tucker) methods.
  - Partial differentiation and total differentiation
  - Excel and IT related tools.
- If you have no idea about these topics, study them immediately, or consider dropping the course.

Textbooks

- There are many good textbooks.
- But, it will take too much time to go over one book.
  - Many textbooks are about 1,000 pages long.
  - It is much more efficient to give lectures without textbooks.
  - I will focus on solving problems.
    - Learning through examples.
- I encourage you to read some famous textbooks on your own (during or after this semester).

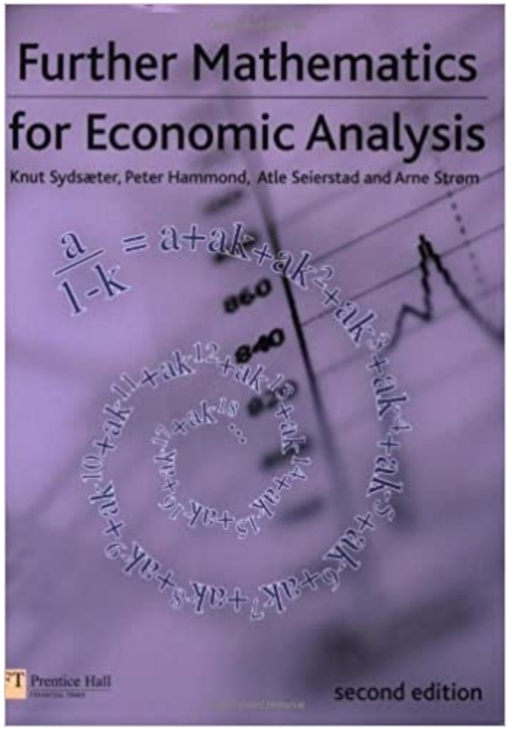
Mathematics for Economists

- Carl Simon & Lawrence Blume, *Mathematics for Economists*, Norton, 1994.
- For advanced undergraduate students and entry-level graduate students.
- Many examples and exercises.
- See if you can solve all the problems in the book.
- My favorite and I highly recommend this book.



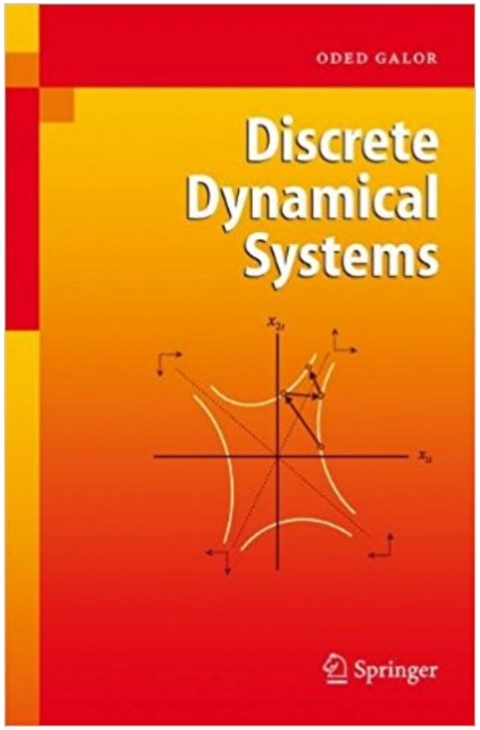
Mathematics for Economists

- Sydsaeter, Hammond, Seierstad, & Strom, *Further Mathematics for Economic Analysis*, 2nd edition, Prentice Hall, 2008.
- Excellent for first-year graduate students.



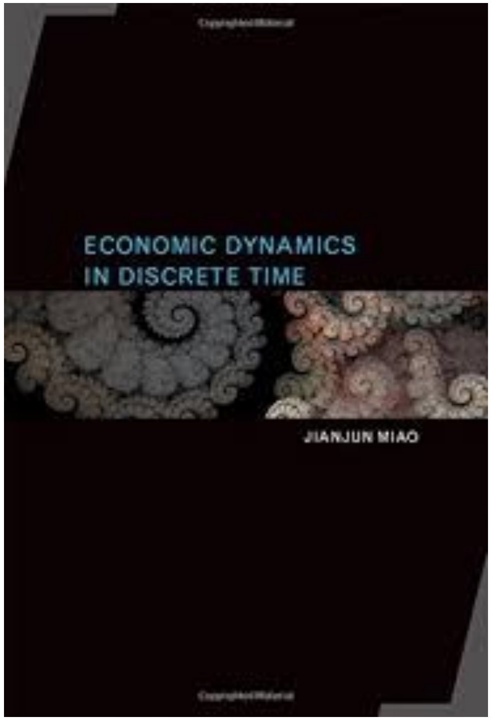
Difference Equations for Macroeconomics

- Oded Galor, *Discrete Dynamical Systems*, Springer, 2010.
- Closely related to the first part of my lecture plan.
- About 150 pages.



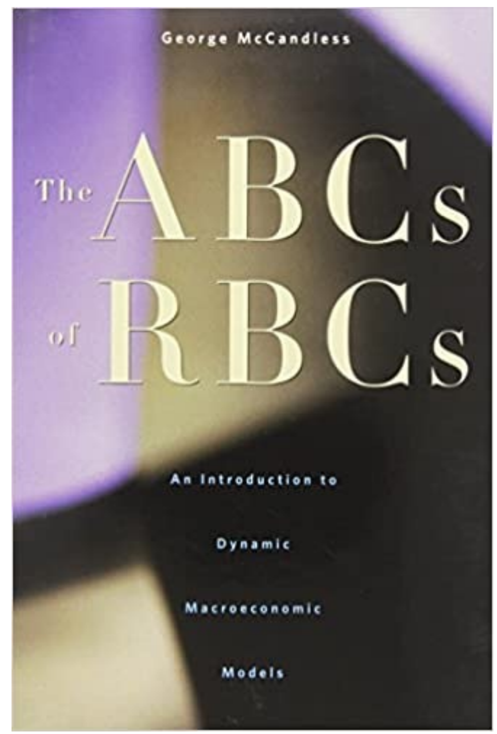
Macroeconomics

- Jianjun Miao, *Economic Dynamics in Discrete Time*, MIT Press, 2014.
- Somewhat related to my lecture plan.
- About 700 pages.



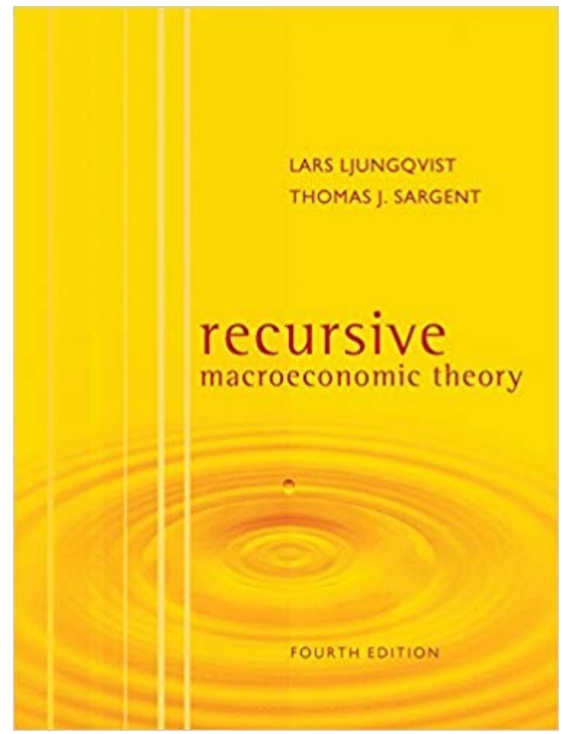
Solving Macro Models

- McCandless, *The ABCs of RBCs: An Introduction to Dynamic Macroeconomic Models*, Harvard University Press, 2008.
- A practical textbook on how to formulate and solve typical macro models.
- Somewhat related to my lecture plan.



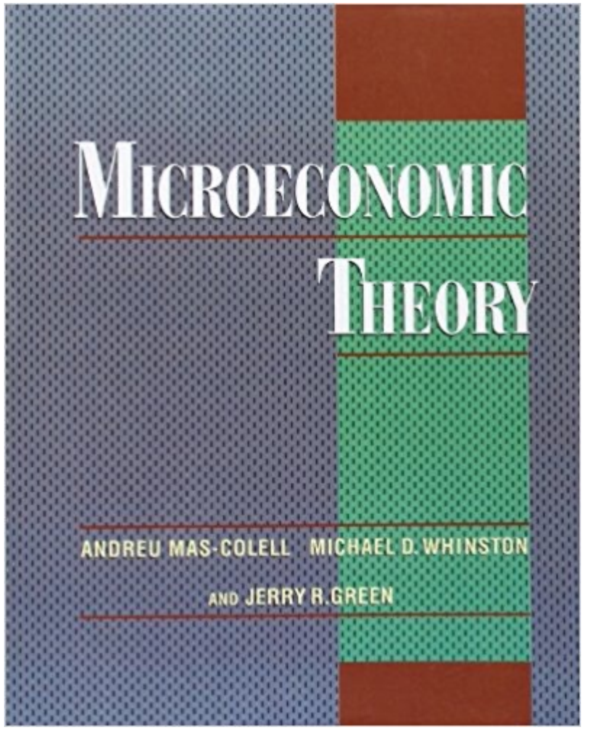
Macroeconomics

- Lars Ljungqvist & Thomas Sargent, *Recursive Macroeconomic Theory*, 4th edition, MIT Press, 2018.
- Extremely famous.
- Starts with dynamic programming.
    - You should wait until the end of this semester.
- Over 1,400 pages!!



Microeconomics

- Andreu Mas-Colell, Michael D. Whinston, & Jerry R. Green, Microeconomic Theory, Oxford University Press, 1995.
- The is the one that all first-year grad students in all grad schools in all countries read. Don't get behind.
- Micro is the foundation of all fields of economics.
- Try to solve all problems in this textbook.



Computer Languages

- 40 years ago, it was OK to write a paper without any computer simulation. Not any more. The old age has gone.
- It is increasingly more important to develop a complicated (realistic) model and solve it using computer.
- It is essential (especially for young students like you) to learn (many) computer languages.

Excel

- Very intuitive and useful.
- We start with Excel to compute dynamic macro models.

Matlab

- Extremely popular among engineering people.
- Especially useful for studying matrix systems.
- The dominant package for simulating macro models.
- Somewhat expensive, but student version is available.

Octave

- **Free** and is a **clone** of Matlab.
  - Octave can execute any Matlab file.
  - Meaning, you do not need to buy Matlab.
  - Slower than Matlab, but don’t complain.

Dynare

- Dynare is a package that assists you simulate your dynamic macro models.
  - It will write Matlab (Octave) cades for you.
- **Dynare runs on Matlab or Octave**.

Mathematica

- Very powerful math package.
- Especially useful for solving nonlinear equations.
- Very popular but very expensive.
- You can execute short mathematica codes at https://www.wolframalpha.com/

Maxima

- A close cousin of Mathematica and this one is **free**.
- This is **not a clone** of Mathematica, but the languages are similar.

Python

- The language of the next generation.
- Available for free.
- For more about this language, please visit https://quantecon.org/

About Assignments

Basic Idea

- My goal is to help you acquire the key analytical methods for macroeconomic research so that you will be able to read professional articles after this semester.
- To acquire new skills, <u>we must focus on solving (many) (time-consuming) questions</u>.

How to Submit Your Work

- All homework assignments must be submitted online at TACT.
  - I will accept PDF files only.
- For many assignments, your answers must be <u>hand-written.</u>
  - Many scanner applications for iPhone/android are available for free.
  - You may also consider using a pen tablet.
  - <u>I strongly suggest that you test each method to see if you can create a PDF of your hand-written letter</u>.

If you have absolutely no idea…

- Try one of the following:

1. My favorite is Microsoft OneNote, which is free and available under many environments.
   - I find it particularly useful on iPad with Apple Pencil.
2. My favorite scanner app is Scannable (by Evernote), but there are many apps.
   - Do not use your photo apps as they create a gigantic PDF file. File size matters.
3. Bring your hand-written notes to a convenience store to manually create a PDF file using a multi-purpose printer there. Costly, not recommended.

Assignment (not to be graded)

- Create a PDF file with your hand-written note.
- Try to find a way to minimize your file size. A PDF from your iPhone camera has a large file size.

Linear Scalar Difference Equations

Varieties of Difference Equations

- Given a function f, suppose x_t evolves according to
$$x_{t+1} = f(x_t)$$
- This is called a **first-order** difference equation.
  - Once you know the current value of x, you immediately know the value of x in the next period.
- This equation is said to be **autonomous**.
  - **Linear** autonomous equation: $x_{t+1} = ax_t + b$ (a, b are parameters)
  - An example of **non-autonomous** equation: $x_{t+1} = 2x_t + t$ (it depends explicitly on t)

Varieties of Difference Equations

- Consider $x_{t+1} = ax_t^2 + b$. This is an example of (first-order) **nonlinear** difference equation.
- Consider $x_{t+1} = ax_t + bx_{t-1}$. This is an example of **second-order** (linear) difference equation.
- Consider $x_{t+1} = ax_t$. This is an example of **homogeneous** difference equation.
  - This is the simplest difference equation.

Solving $x_{t+1} = ax_t$

- Let us start with a homogeneous equation, which is the simplest case.
- Example) $a = 0.5$.
- Thus, $x_{t+1} = 0.5x_t$.
- Suppose that you know the initial value, $x_0 = 1$.
- Then, what is x_1?

Solving $x_{t+1} = ax_t$

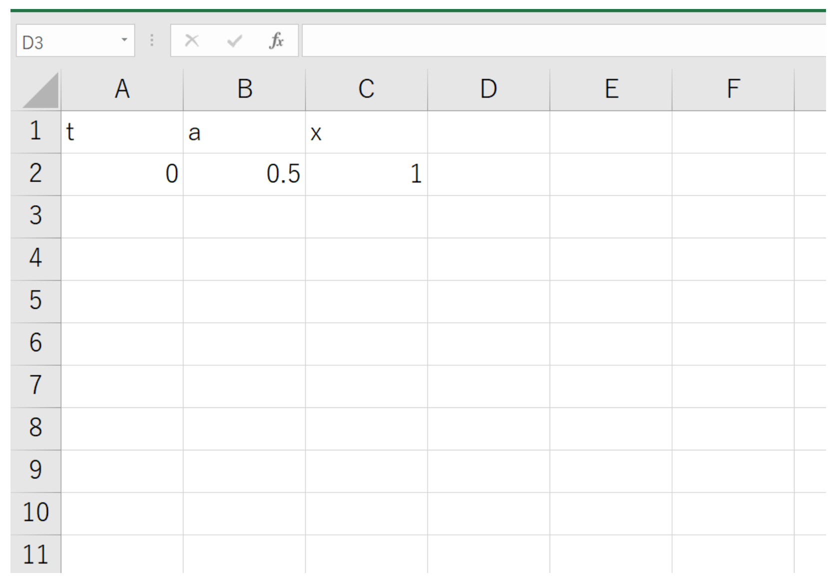
- Substitute $t = 0$ into $x_{t+1} = 0.5x_t$ to obtain
$$x_1 = 0.5 \times x_0 = 0.5 \times 1 = 0.5.$$
- What is x_2 then?
- What is x_t ?

Solving $x_{t+1} = ax_t$

- It is now clear that given the initial condition (i.e., the value of x_0), the difference equation $x_{t+1} = 0.5x_t$ generates a sequence of numbers.
- Because the length this sequence is infinite, it is hopeless to write down the sequence by paper and pencil.
- Let us use Excel (or the like) to continue.

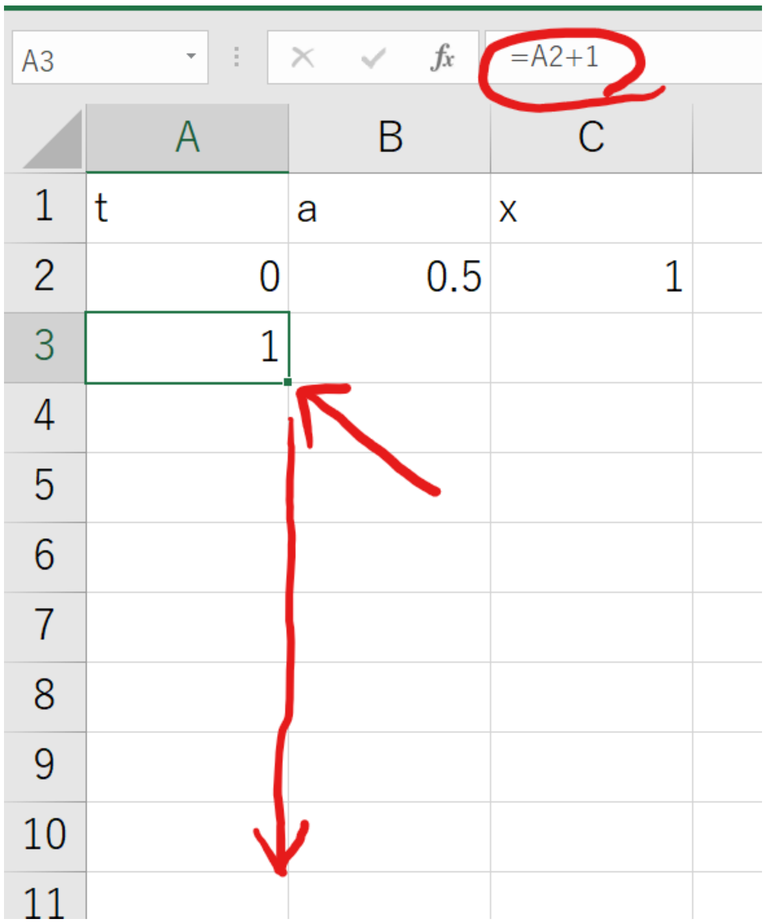
Solving $x_{t+1} = ax_t$

- Column A = period (t)
- Column B = value of a
- Column C = value of x_t
- Cell A2 = 0
- Cell B2 = 0.5 $= a$
- Cell C2 = 1 $= x_0$
- You do not have to fill in all the cells manually.



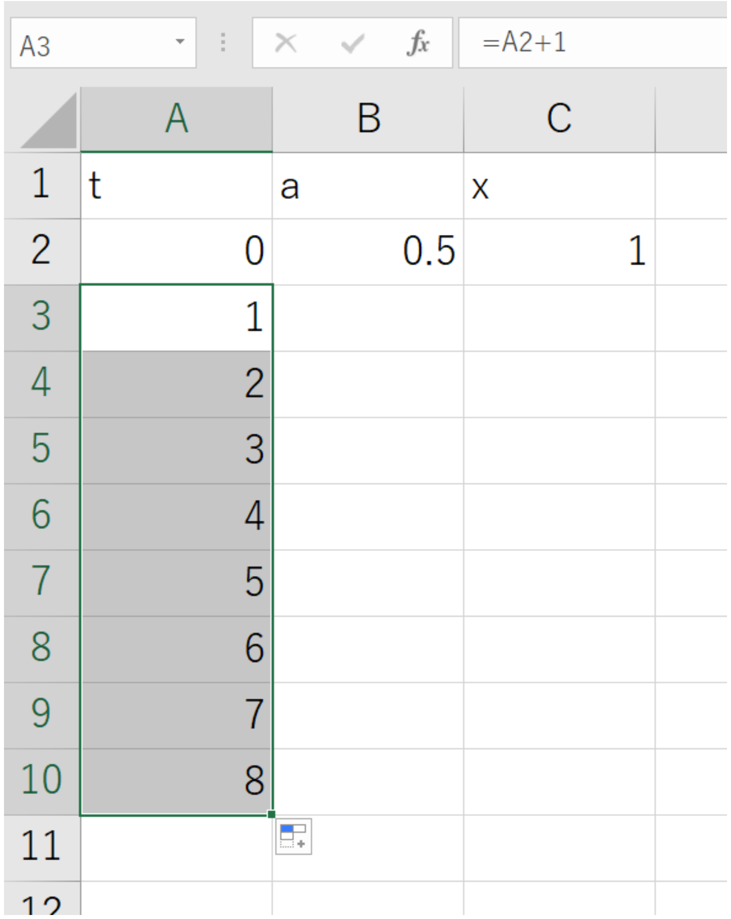
Solving $x_{t+1} = ax_t$

- Cell A3 is "=A2+1"
- Drag the corner of A3 down to A10, say, to see what happens.



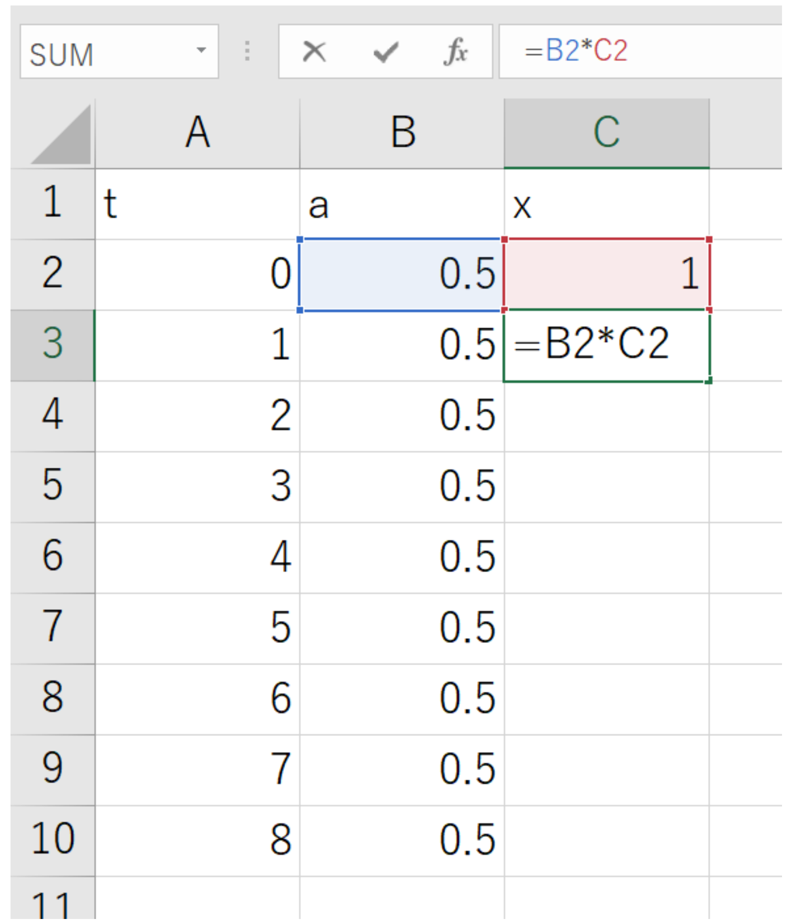
Solving $x_{t+1} = ax_t$

- Cells from A3 to A10 are filled automatically.
- Do the same for column B?
- No, there a better way.
- At B2 cell, <u>double-click on the corner button</u>.



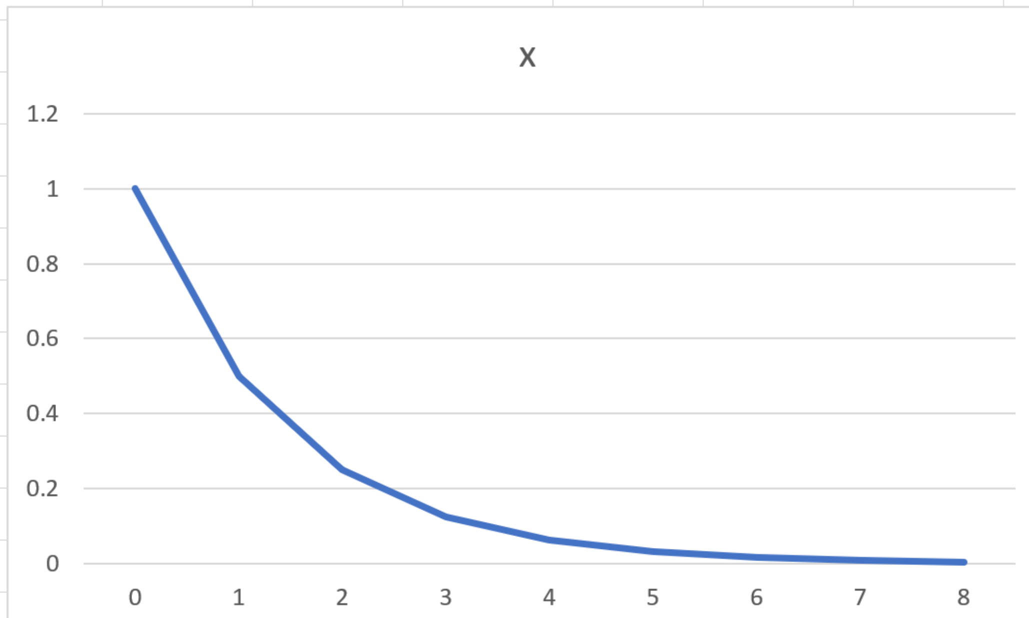
Solving $x_{t+1} = ax_t$

- Cell C3 is “=B2*C2”
- Finally, double click on the button at the corner of C3 cell.



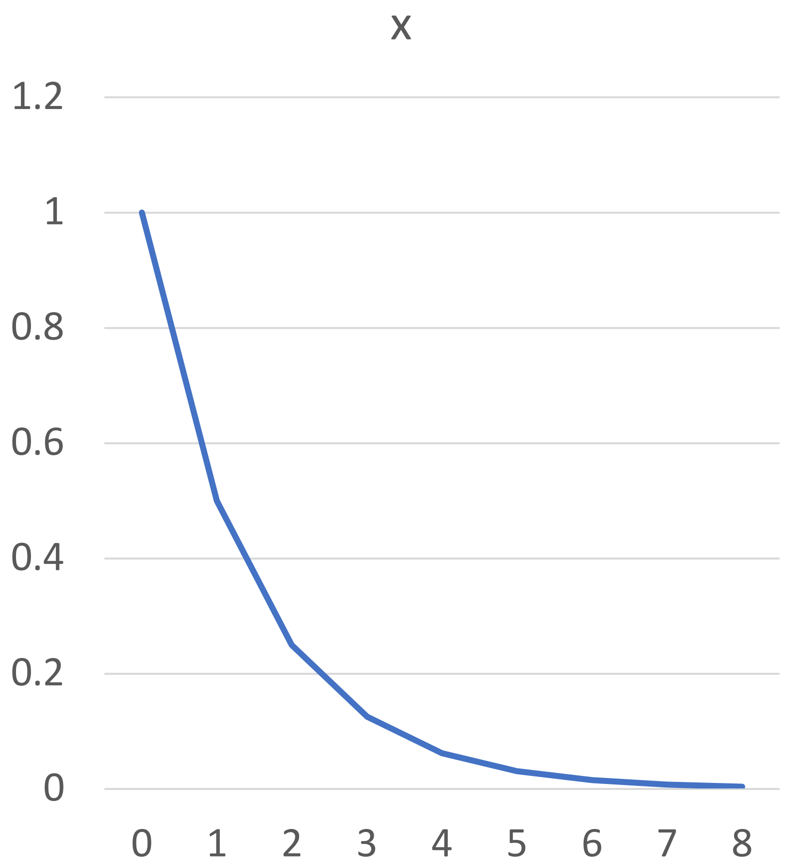
Solving $x_{t+1} = ax_t$

t	a	x
0	0.5	1
1	0.5	0.5
2	0.5	0.25
3	0.5	0.125
4	0.5	0.0625
5	0.5	0.03125
6	0.5	0.015625
7	0.5	0.007813
8	0.5	0.003906



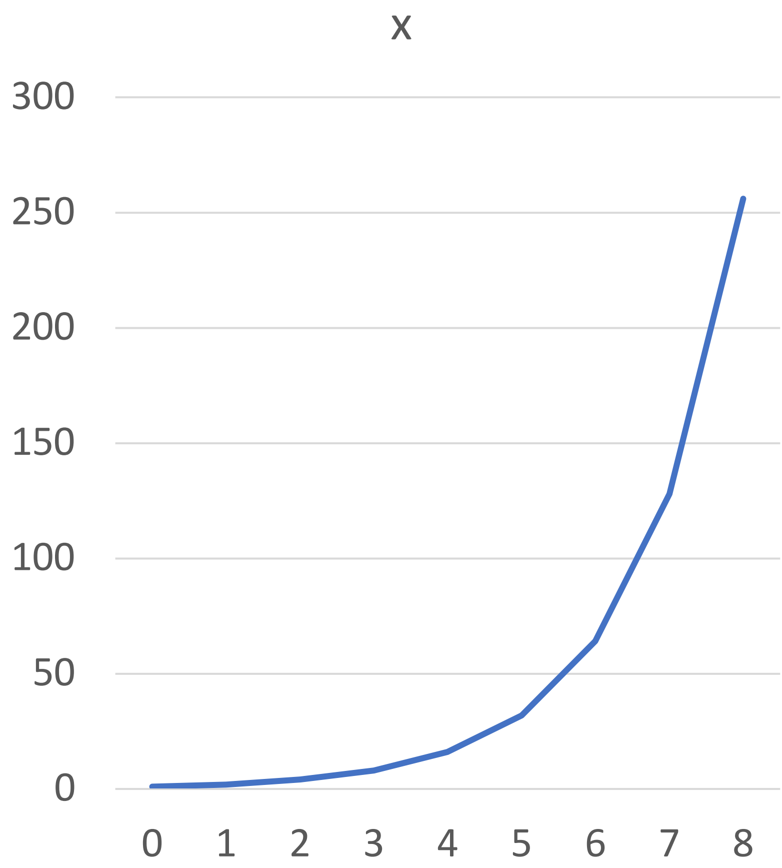
Solving $x_{t+1} = ax_t$

- When $0 < a < 1$, the trajectory exhibits **monotone convergence** to $x = 0$.



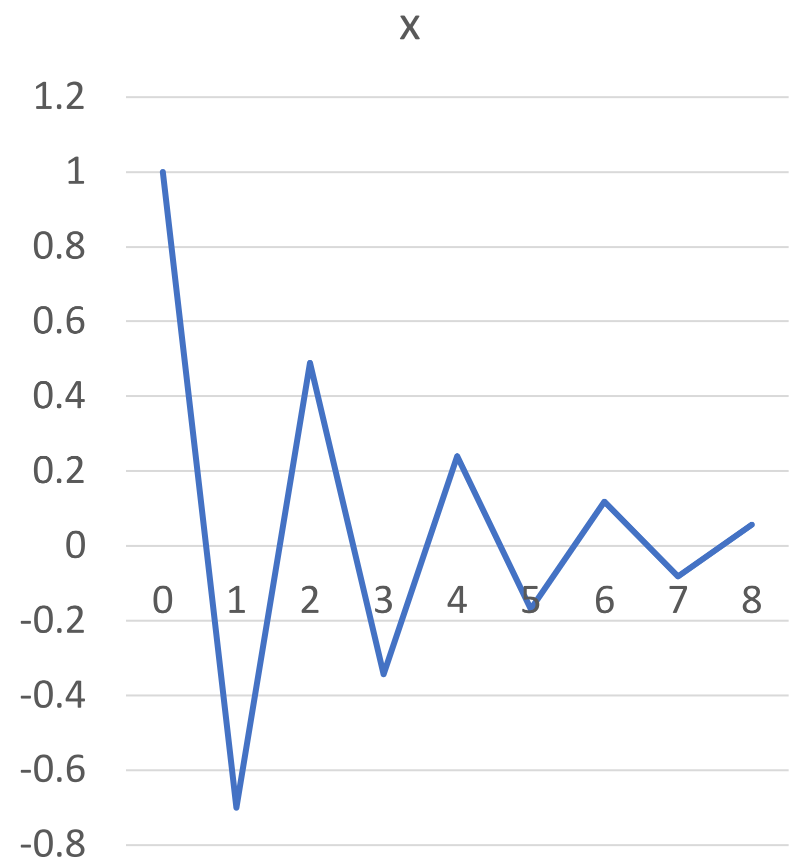
Solving $x_{t+1} = ax_t$

- When $a > 0$, the trajectory exhibits **monotone divergence**.
- The path is said to be **explosive**.



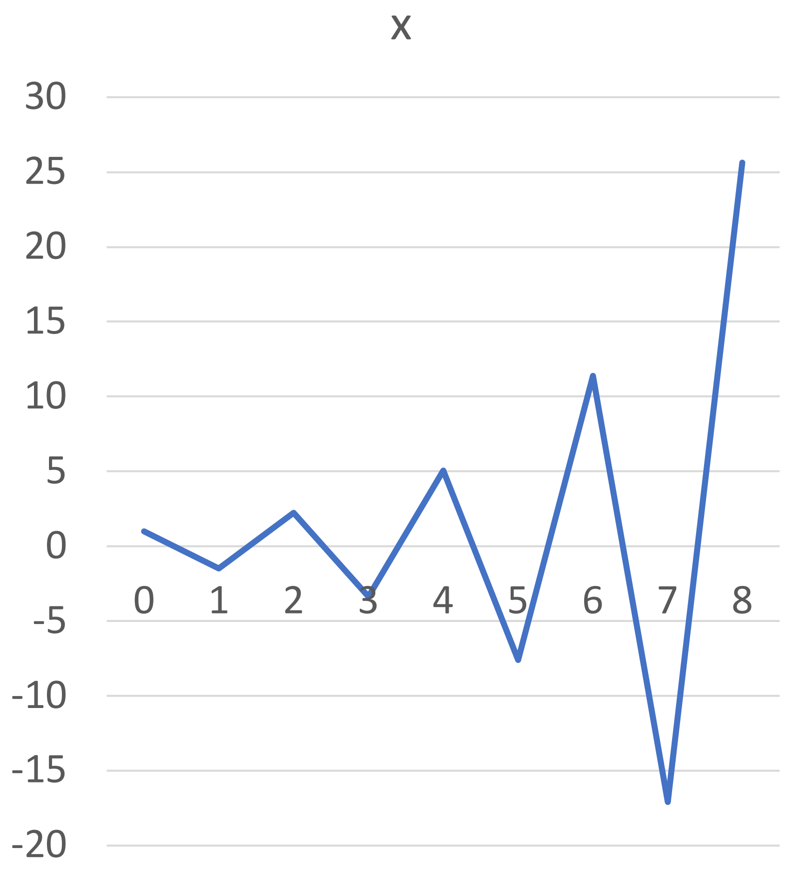
Solving $x_{t+1} = ax_t$

- When $-1 < a < 0$, the trajectory exhibits **damped oscillations**.
- This is convergent.



Solving $x_{t+1} = ax_t$

- When $a < -1$, the trajectory exhibits **explosive oscillations**.



Observations

- Consider $x_{t+1} = f(x_t)$.
- Given a value of x_0 (called the initial condition), we obtain an infinite sequence of numbers satisfying the difference equation.
- The sequence $\{x_t\}_{t=1}^{\infty}$ is the **solution** to the difference equation.
- Two types of solution:
  - **Numerical solution** = computer-generated numbers.
  - **Analytical solution** = a function that reduces the equation to an identity.

Solving $x_{t+1} = ax_t$ Analytically

- Consider $x_{t+1} = ax_t$ with initial condition x_0.
  - This means that a and x_0 are parameters.
- $x_1 = ax_0$. This a number.
- $x_2 = ax_1 = a(ax_0) = a^2 x_0$. Again, a number.
- $x_3 = ax_2 = a(a^2 x_0) = a^3 x_0$.
- We can now guess that

$$x_t = a^t x_0$$

- This is a function that reduces the equation to an identity, so this must be the solution!

Solving $x_{t+1} = ax_t$ Analytically

- How can we verify that $x_t = a^t x_0$ is the solution?
- To see if $x_t = a^t x_0$ solves $x_{t+1} = ax_t$, let us check whether $x_{t+1} - ax_t$ is zero or not:
$$x_{t+1} - ax_t = a^{t+1}x_0 - a \times a^t x_0 = 0.$$
- Thus, **the solution to $\boldsymbol{x_{t+1} = ax_t}$ given $\boldsymbol{x_0}$ is**
$$\boldsymbol{x_t = a^t x_0}$$
- <u>It is extremely important that you remember (and never forget) this result, as we will come back to this many times.</u>

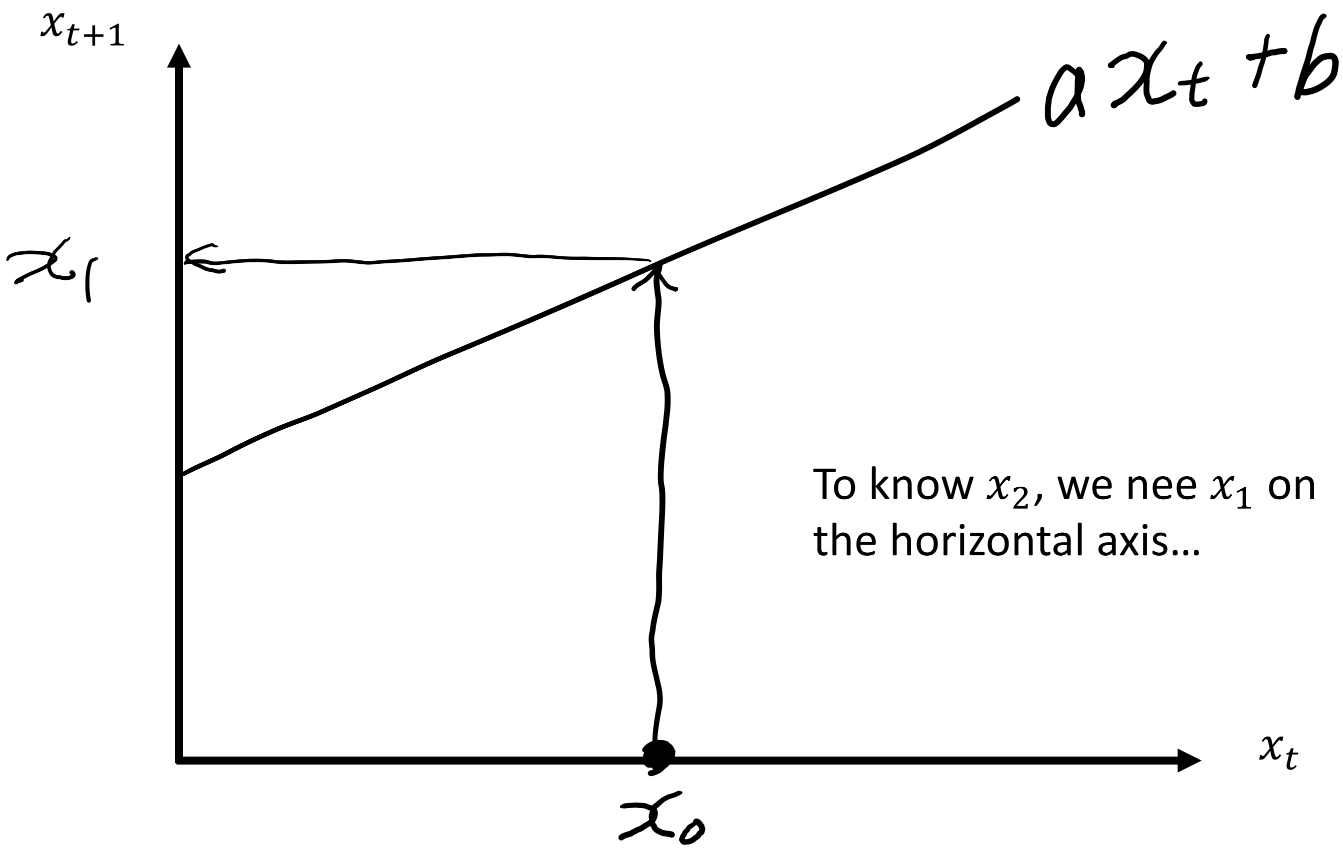
Solving $x_{t+1} = ax_t + b$

- Now consider the general first-order linear scalar difference equation:

$$x_{t+1} = ax_t + b$$

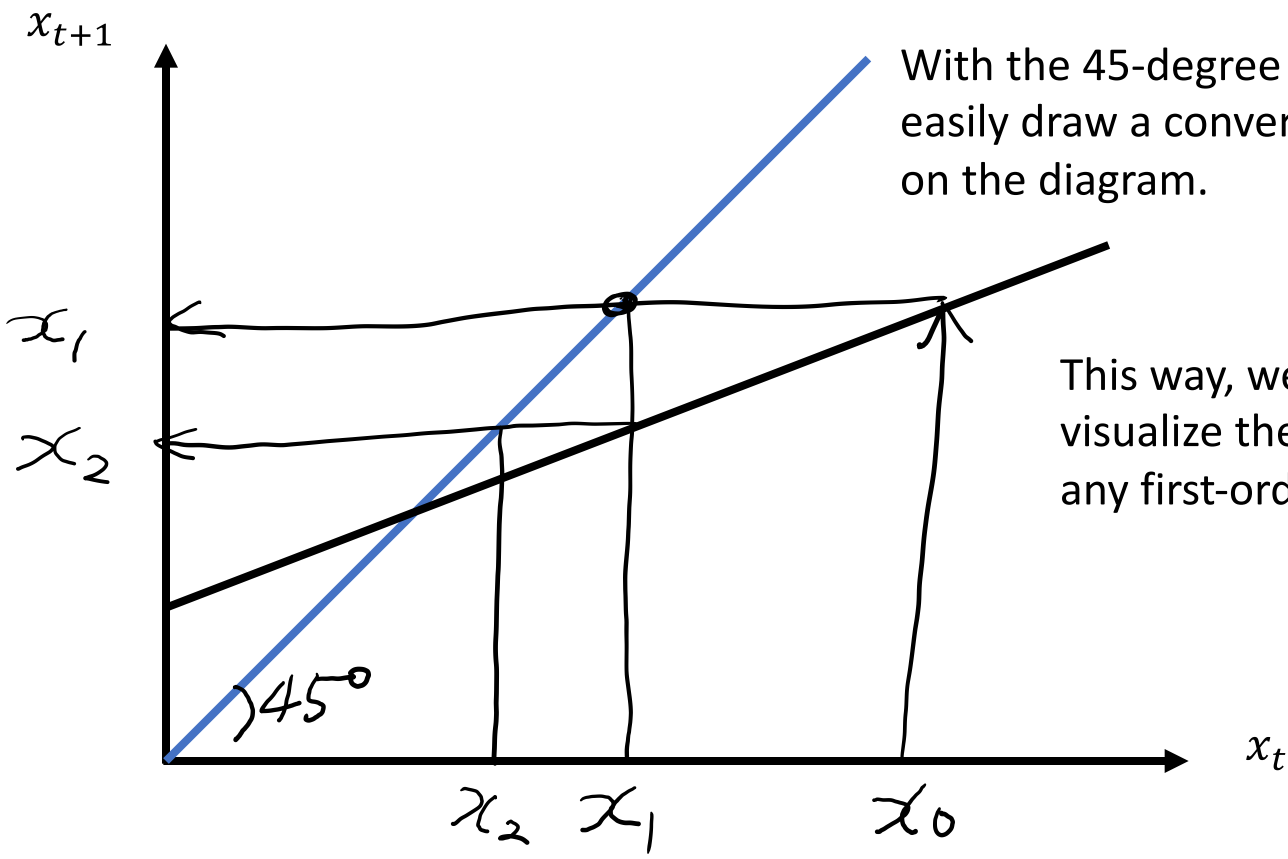
- Suppose $0 < a < 1$ and $b > 0$.
- Let us now draw a diagram.
- x_{t+1} is on the vertical axis.
- x_t is on the horizontal axis.

Solving $x_{t+1} = ax_t + b$



To know x_2, we nee x_1 on the horizontal axis…

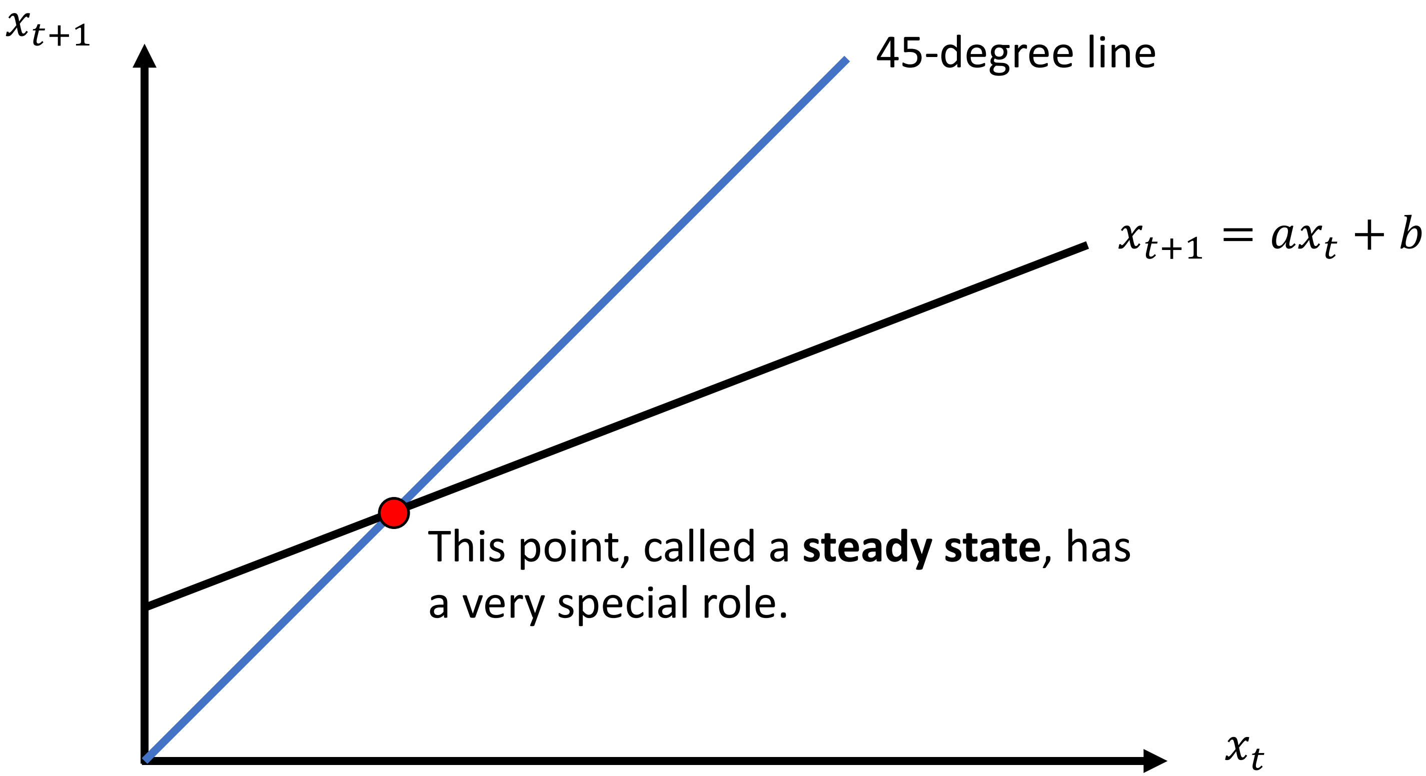
Solving $x_{t+1} = ax_t + b$



With the 45-degree line, we can easily draw a convergent sequence on the diagram.

This way, we can visualize the solution to any first-order DE.

Solving $x_{t+1} = ax_t + b$



Solving $x_{t+1} = ax_t + b$

- Definition: A **steady state** (or, stationary state) of a dynamical system is a solution to the difference equation such that $x_t = x_{ss}$ for all t.
  - x_{ss} is a number.
  - Notation can be anything, but people prefer $\bar{x}$, x^*, x_{ss}, or simply x.
- Finding a steady state is easy.
- Apply the definition to our DE to obtain

$$x_{ss} = ax_{ss} + b$$

- Note that this is a simple equation.

Solving $x_{t+1} = ax_t + b$

- The solution to $x_{ss} = ax_{ss} + b$ (when $a \neq 1$) is
$$x_{ss} = \frac{b}{1-a}$$
- Note that this is a solution to $x_{t+1} = ax_t + b$ as $x_t = x_{ss}$ for all t satisfies the equation.

Solving $x_{t+1} = ax_t + b$

- Consider $x_{t+1} = ax_t + b$.
- How can we solve it analytically?
- **Basic idea: Transform the equation to the one we know the solution**.
- Let us subtract x_{ss} from both sides of the equation (Remember that $x_{ss} = ax_{ss} + b$):

$$\begin{aligned} x_{t+1} - x_{ss} &= ax_t + b - x_{ss} \\ &= ax_t + b - (ax_{ss} - b) \\ &= a(x_t - x_{ss}) \end{aligned}$$

Solving $x_{t+1} = ax_t + b$

- Now we have:

$$x_{t+1} - x_{ss} = a(x_t - x_{ss})$$

- Define

$$\hat{x}_t = x_t - x_{ss}$$

- Then, our non-homogeneous DE reduces to a homogeneous DE:

$$\hat{x}_{t+1} = a\hat{x}_t$$

- What is the solution to this new DE?
- You should be able to solve it within a second.

Solving $x_{t+1} = ax_t + b$

- The solution to $\hat{x}_{t+1} = a\hat{x}_t$ is $\hat{x}_t = a^t\hat{x}_0$.
- Unfortunately, we are not done yet because this is only the solution to the transformed equation, not the original one.
- Our next (and final) step is to recover the solution to the original DE.
- Substitute $\hat{x}_t = x_t - x_{ss}$ back into $\hat{x}_t = a^t\hat{x}_0$ to obtain

$$x_t - x_{ss} = a^t(x_0 - x_{ss})$$

Solving $x_{t+1} = ax_t + b$

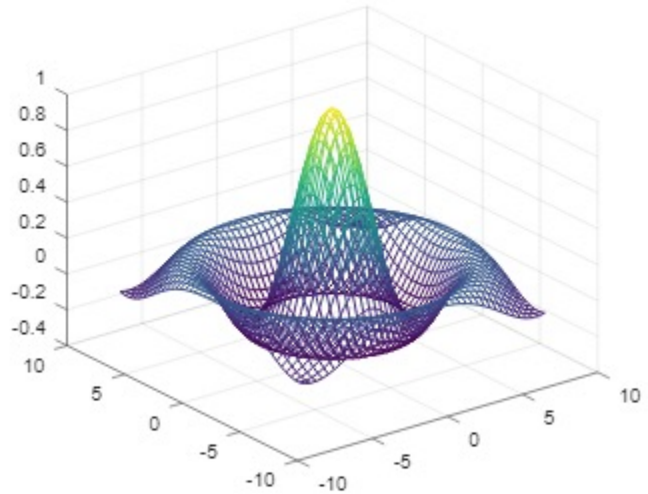
- From our previous expression $x_t - x_{ss} = a^t(x_0 - x_{ss})$, we obtain
$$x_t = x_{ss} + a^t(x_0 - x_{ss})$$
- This is the solution to the original DE.
- When $b = 0$, we have $x_{ss} = 0$ and the solution reduces to
$$x_t = a^t x_0$$

Stability

- Definition: x_{ss} is **stable** if
$$\lim_{t\to\infty} x_t = x_{ss} \text{ for any } x_0$$
- For any value of b,
  - $|a| < 1 \Leftrightarrow x_{ss}$ is stable.
  - $|a| > 1 \Leftrightarrow x_{ss}$ is unstable.
  - $a > 0 \Longrightarrow \{x_t\}_{t=0}^{\infty}$ is monotonic.
  - $a < 0 \Longrightarrow \{x_t\}_{t=0}^{\infty}$ displays oscillations.
- For stability, a plays the central role, while b only determines the level of the steady state.

Get Ready for Octave

- Install Octave on your PC from this site (not the octave site):
- https://www.dynare.org/download/
    - Win: You should choose **Octave 8.1.0**
    - Mac: visit https://www.dynare.org/resources/quick_start/



Get Ready for Python (Optional)

- Install Anaconda (= Python) on your PC.
  - https://python-programming.quantecon.org/getting_started.html
- Or, get a google account.
  - Many of you already have one, I suppose.
- Python is optional, but I recommend that you be multilingual.