Chapter 6

Commutator methods

Let us consider a self-adjoint operator H in a Hilbert space $\mathcal{H}$. As seen in Section 4.4 the resolvent $(H-\lambda\mp i\varepsilon)^{-1}$ does not possess a limit in $\mathscr{B}(\mathcal{H})$ as $\varepsilon\searrow 0$ if $\lambda\in\sigma(H)$. However, the expression $\langle f,(H-\lambda\mp i\varepsilon)^{-1}f\rangle$ may have a limit as $\varepsilon\searrow 0$ for suitable f. In addition, if this limit exists for sufficiently many f, then H is likely to have only absolutely continuous spectrum around λ, see Proposition 4.4.2 for a precise statement.

Our aim in this chapter is to present a method which allows us to determine if the spectrum of H is purely absolutely continuous in some intervals. This method is an extension of Theorem 4.4.3 of Putnam which is valid if both operators H and A are unbounded. Again, the method relies on the positivity of the commutator $[iH,A]$, once this operator is well-defined and localized in the spectrum of H. In fact, it was E. Mourre who understood how the method of Putnam can be sufficiently generalized.

Since the proof of the main result is rather long and technical, we shall first state the main result in a quite general setting. Then, the various tools necessary for understanding this result will be presented, as well as some of its corollaries. Only at the end of the chapter, a proof will be sketched, or presented in a restricted setting.

6.1 Main result

As mentioned above, we will state the main results of the chapter even if it is not fully understandable yet. Additional explanations will be provided in the subsequent sections. Let us however introduce very few information. A self-adjoint operator H in a Hilbert space $\mathcal{H}$ has *a gap* if $\sigma(H)\neq\mathbb{R}$. In the sequel, we shall give a meaning to the requirement "H is of class $C^1(A)$ or of class $C^{1,1}(A)$", but let us mention that the condition H being of class $C^1(A)$, and *a fortiori* of class $C^{1,1}(A)$, ensures that the commutator $[iH,A]$, between two unbounded self-adjoint operators, is well-defined in a sense explained later on.

The following statement corresponds to [ABG, Thm. 7.4.2].

Theorem 6.1.1. *Let H be a self-adjoint operator in $\mathcal{H}$, of class $C^{1,1}(A)$ and having a spectral gap. Let $J\subset\mathbb{R}$ be open and bounded and assume that there exist $a>0$ and*

$K \in \mathscr{K}(\mathcal{H})$ such that

$$E^H(J)[iH,A]E^H(J) \geq aE^H(J) + K.$$

Then H has at most a finite number of eigenvalues in J, multiplicity counted, and has no singular continuous spectrum in J.

In fact, this statement is already a corollary of a more general result that we provide below. For its statement, let us still introduce some information. If A is a second self-adjoint operator in $\mathcal{H}$, with domain $\mathsf{D}(A)$, let us set $\mathcal{G} := \big(\mathsf{D}(A), \mathcal{H}\big)_{1/2,1}$ for the Banach space obtained by interpolation between $\mathsf{D}(A)$ and $\mathcal{H}$ (explained later on). Its dual space is denoted by $\mathcal{G}^*$ and one has $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$ with dense and continuous embeddings, as well as $\mathscr{B}(\mathcal{H}) \subset \mathscr{B}(\mathcal{G}, \mathcal{G}^*)$. If H is of class $C^1(A)$ we also define the subset $\mu^A(H)$ by

$$\mu^A(H) := \big\{\lambda \in \mathbb{R} \mid \exists \varepsilon > 0, a > 0 \text{ s.t. } E(\lambda;\varepsilon)[iH,A]E(\lambda;\varepsilon) \geq aE(\lambda;\varepsilon)\big\}, \tag{6.1}$$

where $E(\lambda;\varepsilon) := E^H\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)$.

The following statement is a slight reformulation of [ABG, Thm 7.4.1].

Theorem 6.1.2. *Let H be a self-adjoint operator in $\mathcal{H}$ and assume that H has a gap and is of class $C^{1,1}(A)$. Then, for each $\lambda \in \mu^A(H)$ the limits $\lim_{\varepsilon \searrow 0}\langle f, (H-\lambda \mp i\varepsilon)^{-1} f\rangle$ exist for any $f \in \mathcal{G}$ and uniformly on each compact subset of $\mu^A(H)$. In particular, if T is a linear operator from $\mathcal{H}$ to an auxiliary Hilbert space, and if T is continuous when $\mathcal{H}$ is equipped with the topology induced by $\mathcal{G}^*$, then T is locally H-smooth on the open set $\mu^A(H)$.*

Note that the notion of H-smooth operator will be introduced later on, but that these operators play an important role for proving the existence and the completeness of some wave operators.

Remark 6.1.3. *In the above two statements, it is assumed that H has a spectral gap, which is a restricting assumption since there also exist operators H with $\sigma(H) = \mathbb{R}$. For example, the operator X of multiplication by the variable in $L^2(\mathbb{R})$ has spectrum equal to $\mathbb{R}$. However, there also exists a version of the Theorems 6.1.1 and 6.1.2 which do not require the existence of a gap. The main interest in the gap assumption is that there exists $\lambda_0 \in \mathbb{R}$ such that $(H-\lambda_0)^{-1}$ is bounded and self-adjoint. This operator can then be used in the proofs and this fact is quite convenient. If such a λ_0 does not exist proofs are a little bit more involved.*

Our main task now is to introduce all the notions such that the above statements become fully understandable.

6.2 Regularity classes

Most of the material presented in this section is borrowed from Chapter 5 of [ABG] to which we refer for more information and for a presentation in a more general setting.

Let us consider a self-adjoint operator A in $\mathcal{H}$ which generates the strongly continuous unitary group $\{\mathrm{e}^{-itA}\}_{t\in\mathbb{R}}$. We also consider a bounded operator S in $\mathcal{H}$. In this setting, the map

$$\mathbb{R} \ni\ t \mapsto \mathscr{U}_t[S] \equiv S(t) := \mathrm{e}^{itA}\, S\, \mathrm{e}^{-itA}\ \in \mathscr{B}(\mathcal{H}) \tag{6.2}$$

is well-defined and its regularity can be studied. In fact, $\mathscr{U}_t : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ defines a weakly continuous representation of $\mathbb{R}$ in $\mathscr{B}(\mathcal{H})$.

We start by some conditions of regularity indexed by a positive integer k.

Definition 6.2.1. *Let $k \in \mathbb{N}$.*

(i) $C^k(A)$ denotes the Banach space of all $S \in \mathscr{B}(\mathcal{H})$ such that the map (6.2) *is k-times strongly continuously differentiable, and endowed with the norm*

$$\|S\|_{C^k} := \Big(\sum_{j=0}^{k} \|S^{(j)}(0)\|^2\Big)^{1/2}, \tag{6.3}$$

where $S^{(j)}(0)$ denotes the j^{th} derivative of $S(t)$ evaluated at $t = 0$.

(ii) $C^k_u(A)$ denotes the Banach space of all $S \in \mathscr{B}(\mathcal{H})$ such that the map (6.2) *is k-times continuously differentiable in norm, and endowed with the norm defined by* (6.3)*.*

It can be shown that these spaces are indeed complete and that $C^k_u(A) \subset C^k(A)$. An equivalent description of the elements of $C^k(A)$ or $C^k_u(A)$ is provided in the following statement, see [ABG, Thm. 5.1.3] for its proof.

Proposition 6.2.2. *Let $k \in \mathbb{N}^*$ and $S \in \mathscr{B}(\mathcal{H})$. Then S belongs to $C^k(A)$ or to $C^k_u(A)$ if and only if $\lim_{t\searrow 0} t^{-k}(\mathscr{U}_t - \mathbf{1})^k[S]$ exists in the strong or in the norm topology of $\mathscr{B}(\mathcal{H})$.*

Note now that a formal computation of $\frac{\mathrm{d}}{\mathrm{d}t}S(t)|_{t=0}$ gives $S'(0) = [iA, S]$. However, this formula has to be taken with some care since it involves the operator A which is often unbounded. In fact, a more precise and alternative description of the $C^1(A)$ condition is often very useful, see [ABG, Lem. 6.2.9] for its proof.

Lemma 6.2.3. *The bounded operator S belongs to $C^1(A)$ if and only if there exists a constant $c < \infty$ such that*

$$\big|\langle Af, iSf\rangle - \langle S^*f, iAf\rangle\big| \le c\|f\|^2, \qquad \forall f \in \mathsf{D}(A). \tag{6.4}$$

In fact, the expression $\langle Af, iSf\rangle - \langle iS^*f, Af\rangle$ defines a quadratic form with domain $\mathsf{D}(A)$, and the condition (6.4) means precisely that this form is bounded. Since $\mathsf{D}(A)$ is dense in $\mathcal{H}$, this form extends to a bounded form on $\mathcal{H}$, and there exists a unique operator in $\mathscr{B}(\mathcal{H})$ which corresponds to this form. For an obvious reason we denote this bounded operator by $[iA, S]$ and the following equality holds for any $f \in \mathsf{D}(A)$

$$\langle Af, iSf\rangle - \langle S^*f, iAf\rangle = \langle f, [iA, S]f\rangle.$$

However, let us stress that the bounded operator $[iA, S]$ has *a priori* no explicit expression on all $f \in \mathcal{H}$.

Let us add some rather simple properties of the class $C^k(A)$ and $C^k_u(A)$.

Proposition 6.2.4. *(i) If S belongs to $C^k(A)$ then $S \in C^{k-1}_u(A)$,*

(ii) If $S, T \in C^k(A)$ then $ST \in C^k(A)$, and if $S, T \in C^k_u(A)$ then $ST \in C^k_u(A)$,

(iii) If S is boundedly invertible, then $S \in C^k(A) \iff S^{-1} \in C^k(A)$, and $S \in C^k_u(A) \iff S^{-1} \in C^k_u(A)$,

(iv) $S \in C^k(A) \iff S^* \in C^k(A)$, and $S \in C^k_u(A) \iff S^* \in C^k_u(A)$.

Exercise 6.2.5. *Provide a proof of the above statements. Note that compared with the general theory presented in [ABG, Sec. 5.1] we are dealing only with a one-parameter unitary group, and only in the Hilbert space $\mathcal{H}$. Multiparameter C_0-group acting on arbitrary Banach spaces are avoided in these notes.*

Let us now present some regularity classes of fractional order. Consider $s \geq 0$, $p \in [0, \infty]$ and let $\ell \in \mathbb{N}$ with $\ell > s$. We can then define

$$\|S\|^{(\ell)}_{s,p} := \|S\| + \Big(\int_{|t|\leq 1} \big\||t|^{-s}(\mathscr{U}_t - \mathbf{1})^\ell[S]\big\|^p \frac{\mathrm{d}t}{|t|}\Big)^{1/p},$$

with the convention that the integral is replaced by a sup when $p = \infty$. It can then be shown that if $\|S\|^{(\ell)}_{s,p} < \infty$ then $\|S\|^{(\ell')}_{s,p} < \infty$ for any integer $\ell' > s$. For that reason, the following definition is meaningful:

Definition 6.2.6. *For any $s \geq 0$ and $p \in [0, \infty]$ we set $C^{s,p}(A)$ for the set of $S \in \mathscr{B}(\mathcal{H})$ such that $\|S\|^{(\ell)}_{s,p} < \infty$ for some (and them for all) $\ell \in \mathbb{N}$ with $\ell > s$. For two different integers $\ell, \ell' > s$, the maps $S \mapsto \|S\|^{(\ell)}_{s,p}$ and $S \mapsto \|S\|^{(\ell')}_{s,p}$ define equivalent norms on $C^{s,p}(A)$. Endowed with any of these norms, $C^{s,p}(A)$ is a Banach space.*

Let us state some relations between these spaces and the spaces $C^k(A)$ and $C^k_u(A)$ introduced above. All these relations are proved in a larger setting in [ABG, Sec. 5.2].

Proposition 6.2.7. *Let $k \in \mathbb{N}$, $s, t \geq 0$ and $p, q \in [1, \infty]$.*

(i) $C^{s,p}(A) \subset C^{t,q}(A)$ if $s > t$ and for p, q arbitrary,

(ii) $C^{t,p}(A) \subset C^{t,q}(A)$ if $q > p$ and in particular for any $p \in (1, \infty)$

$$C^{s,1}(A) \subset C^{s,p}(A) \subset C^{s,\infty}(A), \tag{6.5}$$

(iii) If $s = k$ is an integer one has

$$C^{k,1}(A) \subset C^k_u(A) \subset C^k(A) \subset C^{k,\infty}(A).$$

Note that a very precise formulation of the differences between $C^{k,1}(A)$, $C^k_u(A)$, $C^k(A)$ and $C^{k,\infty}(A)$ is presented in [ABG, Thm. 5.2.6]. Another relation between some of these spaces is also quite convenient:

Proposition 6.2.8. *Let $s \in (0,\infty)$ and $p \in [1,\infty]$, and write $s = k + \sigma$ with $k \in \mathbb{N}$ and $0 < \sigma \leq 1$.*

(i) If $S \in C^{s,p}(A)$ and $j \leq k$, then $S^{(j)}(0) \in C^{s-j,p}(A)$,

(ii) If $\|\cdot\|_{C^{\sigma,p}}$ is one norm on $C^{\sigma,p}(A)$, then

$$\|S\|_{C^k} + \|S^{(k)}(0)\|_{C^{\sigma,p}}$$

defines a norm on $C^{s,p}(A)$. In particular $S \in C^{s,p}(A)$ if and only if $S \in C^k(A)$ and $S^{(k)}(0)$ belongs to $C^{\sigma,p}(A)$.

Relations similar to the one presented in Proposition 6.2.4 also hold in the present context:

Proposition 6.2.9. *(i) If $S,T \in C^{s,p}(A)$, then $ST \in C^{s,p}(A)$,*

(ii) If S is boundedly invertible, then $S \in C^{s,p}(A) \Longleftrightarrow S^{-1} \in C^{s,p}(A)$,

(iii) $S \in C^{s,p}(A) \Longleftrightarrow S^* \in C^{s,p}(A)$.

Let us still mention one more regularity class with respect to A which is quite convenient in applications. The additional continuity condition is related to Dini continuity in classical analysis. For any integer $k \geq 1$ we set $C^{k+0}(A)$ for the set of $S \in C^k(A)$ and such that $S^{(k)}(0)$ satisfies

$$\int_{|t|\leq 1} \big\|(\mathscr{U}_t - \mathbf{1})[S^{(k)}(0)]\big\| \frac{\mathrm{d}t}{|t|} < \infty.$$

Once endowed with the norm

$$\|S\|_{C^{k+0}} := \|S\| + \int_{|t|\leq 1} \big\|(\mathscr{U}_t - \mathbf{1})[S^{(k)}(0)]\big\| \frac{\mathrm{d}t}{|t|}$$

the set $C^{k+0}(A)$ becomes a Banach space and the following relations hold for any $k \in \mathbb{N}$

$$C^{k+0}(A) \subset C^{k,1}(A) \subset C^k_u(A) \subset C^k(A) \subset C^{k,\infty}(A) \tag{6.6}$$

with $C^{0+0}(A) := C^{0,1}(A)$.

Up to now, we have considered only bounded elements S. In the next section we show how these notions can be useful for unbounded operators as well.

6.3 Affiliation

In this section we consider two self-adjoint operators A and H in a Hilbert space $\mathcal{H}$. The various regularity classes introduced before are defined in term of the unitary group generated by A, and the next definition gives a meaning to the regularity of the operator H with respect to A, even if H is unbounded. Before this definition, let us state a simple lemma whose proof depends only on the first resolvent equation and on some analytic continuation argument, see [ABG, Lem. 6.2.1] for the details.

Lemma 6.3.1. *Let $k \in \mathbb{N}$, $s \geq 0$ and $p \in [0, \infty]$, and let H, A be self-adjoint operators in $\mathcal{H}$. Assume that there exists $z_0 \in \mathbb{C}$ such that $(H - z_0)^{-1}$ belongs to $C^k(A)$, $C^k_u(A)$, or to $C^{s,p}(A)$. Then $(H - z)^{-1}$ belongs to the same regularity class for any $z \in \rho(H)$. In addition, if H is bounded, then H itself belongs to the same regularity class.*

The following definition becomes then meaningful.

Definition 6.3.2. *Let $k \in \mathbb{N}$, $s \geq 0$ and $p \in [0, \infty]$, and let H, A be self-adjoint operators in $\mathcal{H}$. We say that H* is of class *$C^k(A)$, $C^k_u(A)$*, or *$C^{s,p}(A)$ if $(H - z)^{-1}$ belongs to such a regularity class for some $z \in \rho(H)$, and thus for all $z \in \rho(H)$.*

Clearly, if the resolvent of H belongs to one of these regularity classes, then the same holds for linear combinations of the resolvent for different values of $z \in \rho(H)$. In fact, by functional calculus one can show that $\eta(H)$ belongs to the same regularity class for suitable function $\eta : \mathbb{R} \to \mathbb{R}$. We state such a result for a rather restricted class of functions η and refer to Theorem 6.2.5 and Corollary 6.2.6 of [ABG] for a more general statement. Note that the proof is rather technical and depends on an explicit formula for the operator $\eta(H)$ in terms of the resolvent of H. For completeness we provide such a formula but omit all the details and the explanations. For suitable functions $\eta : \mathbb{R} \to \mathbb{R}$ the following formula holds:

$$\begin{aligned}\eta(H) = &\sum_{k=0}^{r-1} \frac{1}{\pi k!} \int_{\mathbb{R}} \eta^{(k)}(\lambda) \Im\big(i^k (H - \lambda - i)^{-1}\big) \mathrm{d}\lambda \\ &+ \frac{1}{\pi (r-1)!} \int_0^1 \mu^{r-1} \mathrm{d}\mu \int_{\mathbb{R}} \eta^{(r)}(\lambda) \Im\big(i^r (H - \lambda - i\mu)^{-1}\big) \mathrm{d}\lambda. \end{aligned} \tag{6.7}$$

Proposition 6.3.3. *Assume that H is of class $C^k(A)$, $C^k_u(A)$, or $C^{s,p}(A)$, and let η be a real function belonging to $C^\infty_c(\mathbb{R})$. Then $\eta(H)$ belongs to the same regularity class.*

Let us now mention how the condition *H is of class* $C^1(A)$ can be checked. For a bounded operator H, this has already been mentioned in Lemma 6.2.3. For an unbounded operator H the question is more delicate. We state below a quite technical result. Note that the invariance of the domain of H with respect to the unitary group generated by A is often assumed, and this assumption simplifies quite a lot the argumentation. However, in the following statement such an assumption is not made.

Theorem 6.3.4 (Thm. 6.2.10 of [ABG]). *Let H and A be self-adjoint operators in a Hilbert space $\mathcal{H}$.*

a) The operator H is of class $C^1(A)$ if and only if the following two conditions hold:

(i) There exists $c < \infty$ such that for all $f \in \mathsf{D}(A) \cap \mathsf{D}(H)$

$$\big|\langle Af, Hf\rangle - \langle Hf, Af\rangle\big| \le c\big(\|Hf\|^2 + \|f\|^2\big),$$

(ii) For some $z \in \mathbb{C} \setminus \sigma(H)$ the set

$$\big\{f \in \mathsf{D}(A) \mid (H-z)^{-1} f \in \mathsf{D}(A) \text{ and } (H-\bar{z})^{-1} f \in \mathsf{D}(A)\big\}$$

is a core for A.

b) If H is of class $C^1(A)$, then the $\mathsf{D}(A) \cap \mathsf{D}(H)$ is a core for H and the form $[A, H]$ has a unique extension to a continuous sesquilinear form on $\mathsf{D}(H)$ endowed with the graph topology[1]*. If this extension is still denoted by $[A, H]$, then the following identify holds on $\mathcal{H}$:*

$$[A, (H-z)^{-1}] = -(H-z)^{-1}[A, H](H-z)^{-1}. \tag{6.8}$$

Let us still mention how the equality (6.8) can be understood. If we equip $\mathsf{D}(H)$ with the graph topology (it is then a Banach space), and denote by $\mathsf{D}(H)^*$ its dual space, then one has the following dense inclusions $\mathsf{D}(H) \subset \mathcal{H} \subset \mathsf{D}(H)^*$, and the operator $R(z)$ is bounded from $\mathcal{H}$ to $\mathsf{D}(\mathcal{H})$ and extends to a bounded operator from $\mathsf{D}(H)^*$ to $\mathcal{H}$. Then, the fact that $[A, H]$ has a unique extension to a sesquilinear form on $\mathsf{D}(H)$ means that its continuous extension $[A, H]$ is a bounded operator from $\mathsf{D}(H)$ to $\mathsf{D}(H)^*$. Thus, the r.h.s. of (6.8) corresponds to the product of three bounded operators

$$[A, (H-z)^{-1}] = -\underbrace{(H-z)^{-1}}_{\mathsf{D}(H)^* \to \mathcal{H}} \underbrace{[A, H]}_{\mathsf{D}(H) \to \mathsf{D}(H)^*} \underbrace{(H-z)^{-1}}_{\mathcal{H} \to \mathsf{D}(H)} \tag{6.9}$$

which corresponds to a bounded operator in $\mathcal{H}$. By setting $R(z) := (H-z)^{-1}$, formula (6.9) can also re rewritten as

$$[H, A] = (H-z)[A, R(z)](H-z) \tag{6.10}$$

where the r.h.s. is the product of three bounded operators, namely $(H-z) : \mathsf{D}(H) \to \mathcal{H}$, $[A, R(z)] : \mathcal{H} \to \mathcal{H}$ and $(H-z) : \mathcal{H} \to \mathsf{D}(H)^*$.

Another formula will also be useful later on. For $\tau \neq 0$ let us set $A_\tau := \frac{1}{i\tau}(\mathrm{e}^{i\tau A} - \mathbf{1})$, and observe that if H is of class $C^1(A)$ one has for any $z \in \rho(H)$

$$\begin{aligned}
[A, R(z)] &= s - \lim_{\tau \to 0} \frac{1}{i\tau}\big(\,\mathrm{e}^{i\tau A} R(z)\,\mathrm{e}^{-i\tau A} - R(z)\big) \\
&= s - \lim_{\tau \to 0} \frac{1}{i\tau}\big[\,\mathrm{e}^{i\tau A}, R(z)\big]\,\mathrm{e}^{-i\tau A} = s - \lim_{\tau \to 0}[A_\tau, R(z)]\ .
\end{aligned} \tag{6.11}$$

[1] The graph topology on $\mathsf{D}(H)$ corresponds to the topology obtained by the norm $\|f\|_{\mathsf{D}(H)} = \big(\|f\|^2 + \|Hf\|^2\big)^{1/2}$ for any $f \in \mathsf{D}(H)$.

In relation with formula (6.9) let us observe that if $J \subset \mathbb{R}$ is a bounded Borel set, then $E^H(J)$ is obviously an element of $\mathscr{B}(\mathcal{H})$ but it also belongs to $\mathscr{B}\big(\mathcal{H}, \mathsf{D}(H)\big)$. Indeed, this fact follows from the boundedness of the operator $HE^H(J)$. By duality, it also follows that the operator $E^H(J)$ extends to a bounded operator from $\mathsf{D}(H)^*$ to $\mathcal{H}$, or in short $E^H(J) \in \mathscr{B}\big(\mathsf{D}(H)^*, \mathcal{H}\big)$. This fact is of crucial importance. Indeed, if H is of class $C^1(A)$, then as shown above the operator $[iH, A]$ belongs to $\mathscr{B}\big(\mathsf{D}(H), \mathsf{D}(H)^*\big)$ and therefore the product

$$E^H(J)[iH, A]E^H(J)$$

belongs to $\mathscr{B}(\mathcal{H})$. Such a product was already mentioned in Section 6.1 for the special choice $J = (\lambda - \varepsilon, \lambda + \varepsilon)$ for some fixed $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.

We now introduce an easy result which is often called *the Virial theorem*. Note that this result is often stated without the appropriate assumption.

Proposition 6.3.5. *Let H and A be self-adjoint operator in $\mathcal{H}$ such that H is of class $C^1(A)$. Then $E^H(\{\lambda\})[A, H]E^H(\{\lambda\}) = 0$ for any $\lambda \in \mathbb{R}$. In particular, if f is an eigenvector of H then $\langle f, [A, H]f\rangle = 0$.*

Proof. We must show that if $\lambda \in \mathbb{R}$ and $f_1, f_2 \in \mathsf{D}(H)$ satisfy $Hf_k = \lambda f_k$ for $k = 1, 2$, then $\langle f_1, [A, H]f_2\rangle = 0$. Since $f_1 = (\lambda - i)(H - i)^{-1}f_1$ and $f_2 = (\lambda + i)(H + i)^{-1}f_2$ we get by (6.10) and (6.11) that

$$\begin{aligned}
\langle f_1, [A, H]f_2\rangle &= -(\lambda + i)^2\langle f_1, [A, (H + i)^{-1}]f_2\rangle \\
&= -(\lambda + i)^2 \lim_{\tau \to 0} \Big\{\langle f_1, A_\tau(H + i)^{-1}f_2\rangle - \langle (H - i)^{-1}f_1, A_\tau f_2\rangle\Big\}.
\end{aligned}$$

We finally observe that for $\tau \neq 0$, the term into curly brackets is always equal to 0. □

In relation with the definition of $\mu^A(H)$ mentioned in (6.1) let us still introduce two functions which play an important role in Mourre theory. These functions are well defined if the operator H is of class $C^1(A)$. They provide what is called *the best Mourre estimate*, and the first one is defined for any $\lambda \in \mathbb{R}$ by

$$\varrho_H^A(\lambda) := \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ s.t. } E(\lambda; \varepsilon)[iH, A]E(\lambda; \varepsilon) \geq aE(\lambda; \varepsilon)\big\}.$$

The second function looks similar to the previous one, but the inequality holds modulo a compact operator, namely

$$\begin{aligned}
&\tilde{\varrho}_H^A(\lambda) \\
&:= \sup\big\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ and } K \in \mathscr{K}(\mathcal{H}) \text{ s.t. } E(\lambda; \varepsilon)[iH, A]E(\lambda; \varepsilon) \geq aE(\lambda; \varepsilon) + K\big\}.
\end{aligned}$$

The relation between ϱ_H^A and $\mu^A(H)$ is rather clear. By looking back to the definition of (6.1) one gets

$$\mu^A(H) = \{\lambda \in \mathbb{R} \mid \varrho_H^A(\lambda) > 0\}.$$

Many properties of these functions have been deduced in [ABG, Sec. 7.2].

Proposition 6.3.6. *(i) The function $\varrho_H^A : \mathbb{R} \to (-\infty, \infty]$ is lower semicontinuous and $\varrho_H^A(\lambda) < \infty$ if and only if $\lambda \in \sigma(H)$,*

(ii) The function $\tilde{\varrho}_H^A : \mathbb{R} \to (-\infty, \infty]$ is lower semicontinuous and satisfies $\tilde{\varrho}_H^A \geq \varrho_H^A$. Furthermore $\tilde{\varrho}_H^A(\lambda) < \infty$ if and only if $\lambda \in \sigma_{ess}(H)$.

Let us now state and prove a corollary of the Virial theorem showing that when $\tilde{\varrho}_H^A$ is strictly positive, then only a finite number of eigenvalues can appear.

Corollary 6.3.7. *Let H and A be self-adjoint operator in $\mathcal{H}$ such that H is of class $C^1(A)$. If $\tilde{\varrho}_H^A(\lambda) > 0$ for some $\lambda \in \mathbb{R}$ then λ has a neighbourhood in which there is at most a finite number of eigenvalues of H, each of finite multiplicity.*

Proof. Let $\varepsilon > 0$, $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that

$$E(\lambda;\varepsilon)[iH, A]E(\lambda;\varepsilon) \geq aE(\lambda;\varepsilon) + K. \tag{6.12}$$

If g is an eigenvector of H associated with an eigenvalue in $(\lambda - \varepsilon, \lambda + \varepsilon)$ and if $\|g\| = 1$, then (6.12) and the Virial theorem imply that $\langle g, Kg\rangle < -a$. By contraposition, assume that the statement of the lemma is false. Then there exists an infinite orthogonal sequence $\{g_j\}$ of eigenvectors of H in $E(\lambda;\varepsilon)\mathcal{H}$. In particular, $w - \lim_{j\to\infty} g_j = 0$, as a consequence of the orthogonality of the sequence. However, since K is compact, Kg_j goes strongly to 0 as $j \to \infty$, and then one has $\langle g_j, Kg_j\rangle \to 0$ as $j \to \infty$. This contradicts the inequality $\langle g_j, Kg_j\rangle \leq -a < 0$. □

The previous result can then be used for showing that the two functions ϱ_H^A and $\tilde{\varrho}_H^A$ are in fact very similar. The proof of the following statement can be found in [ABG, Thm. 7.2.13].

Theorem 6.3.8. *Let H and A be self-adjoint operator in $\mathcal{H}$ such that H is of class $C^1(A)$, and let $\lambda \in \mathbb{R}$. If λ is an eigenvalue of H and $\tilde{\varrho}_H^A(\lambda) > 0$, then $\varrho_H^A(\lambda) = 0$. Otherwise $\varrho_H^A(\lambda) = \tilde{\varrho}_H^A(\lambda)$.*

As a conclusion of this section let us mention a result about the stability of the function $\tilde{\varrho}$. Once this stability is proved, the applicability of the previous corollary is quite enlarged.

Theorem 6.3.9. *Let H, H_0 and A be self-adjoint operators in a Hilbert space $\mathcal{H}$ and such that H_0 and H are of class $C_u^1(A)$. If the difference $(H + i)^{-1} - (H_0 + i)^{-1}$ is compact, then $\tilde{\varrho}_H^A(\lambda) = \tilde{\varrho}_{H_0}^A(\lambda)$.*

Note that the content of Proposition 6.3.3 for $\eta(H)$ is necessary for the proof of this statement, and that the $C_u^1(A)$-condition can not be weakened. We refer to [ABG, Thm. 7.2.9] for the details.

6.4 Locally smooth operators

An important ingredient for showing the absence of singular continuous spectrum and for proving the existence and the completeness of some wave operators is the notion of *locally smooth operators*. Such operators were already mentioned in the statement of Theorem 6.1.2 and we shall now provide more information on them. Note that an operator T is always smooth *with respect to another operator*, it is thus a relative notion.

Definition 6.4.1. *Let $J \subset \mathbb{R}$ be an open set, and let $\big(H, \mathsf{D}(H)\big)$ be a self-adjoint operator in $\mathcal{H}$. A linear continuous operator $T : \mathsf{D}(H) \to \mathcal{H}$ is* locally H-smooth on J *if for each compact subset $K \subset J$ there exists a constant $C_K < \infty$ such that*

$$\int_{-\infty}^{\infty} \|T\,\mathrm{e}^{-itH} f\|^2 \mathrm{d}t \leq C_K \|f\|^2, \qquad \forall f \in E^H(K)\mathcal{H}. \tag{6.13}$$

In the next statement, we shall show that this notion can be recast in a time-independent framework. However some preliminary observations are necessary. As already mentioned in the previous section, by endowing $\mathsf{D}(H)$ with its graph norm, one gets the continuous and dense embeddings

$$\mathsf{D}(H) \subset \mathcal{H} \subset \mathsf{D}(H)^*,$$

and a continuous extension of $R(z) \equiv (H-z)^{-1} \in \mathscr{B}\big(\mathcal{H}, \mathsf{D}(H)\big)$ to an element $R(z) \in \mathscr{B}\big(\mathsf{D}(H)^*, \mathcal{H}\big)$, for any $z \in \rho(H)$. By choosing $z = \lambda + i\mu$ with $\mu > 0$ one observes that

$$\begin{aligned}\delta_{(\mu)}(H-\lambda) &:= \frac{1}{\pi}\Im R(\lambda+i\mu)\\ &= \frac{1}{2\pi i}\big(R(\lambda+i\mu) - R(\lambda-i\mu)\big) = \frac{\mu}{\pi} R(\lambda \mp i\mu) R(\lambda \pm i\mu)\end{aligned}$$

and infers that $\delta_{(\mu)}(H-\lambda) \in \mathscr{B}\big(\mathsf{D}(H)^*, \mathsf{D}(H)\big)$. Therefore, if $T \in \mathscr{B}\big(\mathsf{D}(H), \mathcal{H}\big)$ then $TR(z) \in \mathscr{B}(\mathcal{H})$ and $R(z)T^* = \big(TR(\bar{z})\big)^* \in \mathscr{B}(\mathcal{H})$, and consequently $T\delta_{(\mu)}(H-\lambda)T^* \in \mathscr{B}(\mathcal{H})$. Finally, by the C^*-property $\|S^*S\| = \|S^*\|^2 = \|S\|^2$ one deduces that

$$\|T\delta_{(\mu)}(H-\lambda)T^*\| = \frac{\mu}{\pi}\|TR(\lambda \mp i\mu)\|^2 = \frac{\mu}{\pi}\|R(\lambda \pm i\mu)T^*\|^2.$$

We are now ready to state:

Proposition 6.4.2. *A linear continuous operator $T : \mathsf{D}(H) \to \mathcal{H}$ is locally H-smooth on an open set J if and only if for each compact subset $K \subset J$ there exists a constant $C_K < \infty$ such that*

$$\big\|T \Im R(z) T^*\big\| \leq C_K \qquad \text{if } \Re(z) \in K \text{ and } 0 < \Im(z) < 1. \tag{6.14}$$

The proof of this statement is not difficult but a little bit too long. The main ingredients are the equalities

$$R(\lambda \pm i\mu) = i \int_0^{\pm\infty} \mathrm{e}^{it\lambda}\,\mathrm{e}^{-itH-\mu|t|}\,\mathrm{d}t$$

and

$$\delta_{(\mu)}(H-\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{it\lambda}\,\mathrm{e}^{-itH-\mu|t|}\,\mathrm{d}t$$

which were already mentioned in Remark 5.1.2.

Exercise 6.4.3. *Provide the proof of the above statement, which can be borrowed from [ABG, Prop. 7.1.1].*

We shall immediately provide a statement which shows the importance of this notion of local smoothness for the existence and the completeness of the wave operators.

Theorem 6.4.4. *Let H_1, H_2 be two self-adjoint operators in a Hilbert space $\mathcal{H}$, with spectral measure denoted respectively by E_1 and E_2. Assume that for $j \in \{1,2\}$ there exist $T_j \in \mathscr{B}\big(\mathsf{D}(H_j),\mathcal{H}\big)$ which satisfy*

$$\langle H_1 f_1, f_2\rangle - \langle f_1, H_2 f_2\rangle = \langle T_1 f_1, T_2 f_2\rangle \qquad \forall f_j \in \mathsf{D}(H_j).$$

If in addition there exists an open set $J \subset \mathbb{R}$ such that T_j are locally H_j-smooth on J, then

$$W_\pm\big(H_1,H_2,E_2(J)\big) := s-\lim_{t\to\pm\infty} \mathrm{e}^{itH_1}\,\mathrm{e}^{-itH_2}\,E_2(J) \tag{6.15}$$

exist and are bijective isometries of $E_2(J)\mathcal{H}$ onto $E_1(J)\mathcal{H}$.

Proof. The existence of the limits (6.15) is a simple consequence of the following assertion: for any $f_2 \in \mathcal{H}$ such that $E_2(K_2)f_2 = f_2$ for some compact set $K_2 \subset J$, and for any $\theta_1 \in C_c^\infty(J)$ with $\theta_1(x) = 1$ for any x in a neighbourhood of K_2 the limit

$$s-\lim_{t\to\pm\infty} \theta_1(H_1)\,\mathrm{e}^{itH_1}\,\mathrm{e}^{-itH_2}\,f_2 \tag{6.16}$$

exists, and

$$s-\lim_{t\to\pm\infty} \big[1-\theta_1(H_1)\big]\,\mathrm{e}^{itH_1}\,\mathrm{e}^{-itH_2}\,f_2 = 0. \tag{6.17}$$

Let us now prove (6.16). For that purpose we set $W(t) := \theta_1(H_1)\,\mathrm{e}^{itH_1}\,\mathrm{e}^{-itH_2}$ and observe that for any $f_1 \in \mathcal{H}$ and $s < t$:

$$\begin{aligned}
|\langle f_1, [W(t)-W(s)]f_2\rangle| &= \Big|\int_s^t \langle T_1\,\mathrm{e}^{-i\tau H_1}\,\theta_1(H_1)f_1, T_2\,\mathrm{e}^{-i\tau H_2}\,f_2\rangle \mathrm{d}\tau\Big| \\
&\leq \Big[\int_s^t \|T_1\,\mathrm{e}^{-i\tau H_1}\,\theta_1(H_1)f_1\|^2\mathrm{d}\tau\Big]^{1/2}\Big[\int_s^t \|T_2\,\mathrm{e}^{-i\tau H_2}\,f_2\|^2\mathrm{d}\tau\Big]^{1/2} \\
&\leq C_{K_1}\|f\|\Big[\int_s^t \|T_2\,\mathrm{e}^{-i\tau H_2}\,f_2\|^2\mathrm{d}\tau\Big]^{1/2}
\end{aligned}$$

with $K_1 = \operatorname{supp}\theta_1$. We thus obtain that $\|[W(t)-W(s)]f_2\| \to 0$ as $s \to \infty$ or $t \to -\infty$, which proves (6.16).

For the proof of (6.17) let $\theta_2 \in C_c^\infty(J)$ with $\theta_2(x) = 1$ if $x \in K_2$ and such that $\theta_1\theta_2 = \theta_2$. Then $f_2 = \theta_2(H_2)f_2$ and $[1-\theta_1(H_1)]\theta_2(H_2) = [1-\theta_1(H_1)][\theta_2(H_2)-\theta_2(H_1)]$. Hence (6.17) follows from

$$s - \lim_{|t|\to\infty} \big\|[\theta_2(H_2)-\theta_2(H_1)]\,\mathrm{e}^{-itH_2}\, f_2\big\| = 0. \tag{6.18}$$

In fact, we shall prove this estimate for any $\theta_2 \in C_0(\mathbb{R})$. Let us set $r_z(x) := (x-z)^{-1}$ for any $x \in \mathbb{R}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. Since the vector space generated by the family of functions $\{r_z\}_{z\in\mathbb{C}\setminus\mathbb{R}}$ is a dense subset of $C_0(\mathbb{R})$ it is enough to show (6.18) with θ_2 replaced by r_z for any $z \in \mathbb{C}\setminus\mathbb{R}$. Set $R_j = (H_j - z)^{-1}$ for some fixed $z \in \mathbb{C}\setminus\mathbb{R}$ and observe that for any $g_j \in \mathcal{H}$

$$\begin{aligned}
|\langle g_1, (R_1 - R_2)g_2\rangle| &= |\langle R_1^* g_1, H_2R_2g_2\rangle - \langle H_1R_1^*g_1, R_2g_2\rangle| \\
&= |\langle T_1R_1^*g_1, T_2R_2g_2\rangle| \\
&\le \|T_1R_1^*\|\,\|g_1\|\,\|T_2R_2g_2\|.
\end{aligned}$$

Taking $g_2 = \mathrm{e}^{-itH_2}\, f_2$ we see that it is enough to prove that $\|T_2R_2\,\mathrm{e}^{-itH_2}\, f_2\| \to 0$ as $|t| \to \infty$. But this is an easy consequence of the fact that both the function $F(t) := T_2R_2\,\mathrm{e}^{-itH_2}\, f_2$ and its derivative are square integrable on $\mathbb{R}$.

As a consequence of the previous arguments, we have thus obtained that (6.16) exists. Clearly the same arguments apply for the existence of $W_\pm\big(H_2, H_1, E_1(J)\big)$. It then follows that $W_\pm\big(H_2,H_1,E_1(J)\big) = W_\pm\big(H_1,H_2,E_2(J)\big)^*$ from which one deduces the final statement, see also Proposition 5.2.3. □

6.5 Limiting absorption principle

Since the utility of locally smooth operators has been illustrated in the previous theorem, it remains to show how such operators can be exhibited. In this section, we provide this kind of information, and start with the so-called *limiting absorption principle*.

By looking back to the equation (6.14) and by assuming that $T \in \mathscr{B}(\mathcal{H})$, one observes that the main point is to obtain an inequality of the form

$$\big|\big\langle f, \Im R(\lambda+i\mu)f\big\rangle\big| \le C_K\|f\|_{\mathcal{G}}^2 \tag{6.19}$$

for some compact set $K \subset \mathbb{R}$, all $\lambda \in K$ and $\mu > 0$. Note that we have used the notation $\mathcal{G} := T^*\mathcal{H}$ endowed with the norm $\|f\|_{\mathcal{G}} := \inf\{\|g\| \mid T^*g = f\}$ for any $f \in \mathcal{G}$.

Let us be a little bit more general. Consider any Banach space $\mathcal{G}$ such that $\mathcal{G} \subset \mathsf{D}(H)^*$ continuously and densely. By duality it implies the existence of a continuous embedding $\mathsf{D}(H) \subset \mathcal{G}^*$, but this embedding is not dense in general. The closure of $\mathsf{D}(H)$ inside $\mathcal{G}^*$ is thus denoted by $\mathcal{G}^{*\circ}$, and is equipped with the Banach space structure inherited from $\mathcal{G}^*$. It then follows that $\mathscr{B}\big(\mathsf{D}(H)^*,\mathsf{D}(H)\big) \subset \mathscr{B}(\mathcal{G},\mathcal{G}^{*\circ}) \subset \mathscr{B}(\mathcal{G},\mathcal{G}^*)$ and in particular

$$\Im R(z) \in \mathscr{B}\big(\mathsf{D}(H)^*,\mathsf{D}(H)\big) \subset \mathscr{B}(\mathcal{G},\mathcal{G}^{*\circ}) \subset \mathscr{B}(\mathcal{G},\mathcal{G}^*), \qquad \forall z \in \rho(H).$$

Definition 6.5.1. *Let J be an open set, H a self-adjoint operator and $\mathcal{G} \subset \mathsf{D}(H)^*$.*

i) The generalized limiting absorption holds for H in $\mathcal{G}$ and locally on J *if for each compact subset $K \subset J$ there exists $C_K < \infty$ such that* (6.19) *holds for all $f \in \mathcal{G}$, any $\lambda \in K$ and $\mu > 0$, or equivalently if*

$$\sup_{\lambda\in K,\mu>0} \|\Im R(\lambda + i\mu)\|_{\mathscr{B}(\mathcal{G},\mathcal{G}^*)} < \infty$$

for any compact subset $K \subset J$.

ii) The strong generalized limiting absorption holds for H in $\mathcal{G}$ and locally on J *if*

$$\lim_{\mu\searrow 0}\langle f, \Im R(\lambda + i\mu)f\rangle =: \langle f, \Im R(\lambda + i0)f\rangle$$

exists for any $\lambda \in J$ and $f \in \mathcal{G}$, uniformly in λ on any compact subset of J.

Note that by an application of the uniform boundedness principle the generalized limiting absorption (GLAP) holds if the strong GLAP is satisfied. In the next statement we shall make the link between GLAP and locally smooth operators. For that purpose, we first recall a consequence of Stone's formula, see Proposition 4.4.1:

$$E\big((a,b)\big) + \frac{1}{2}E(\{a\}) + \frac{1}{2}E(\{b\}) = w - \lim_{\mu\searrow 0}\int_a^b \delta_{(\mu)}(H-\lambda)\mathrm{d}\lambda. \tag{6.20}$$

Namely, for any $a < b$ and $f \in \mathcal{H}$ one has

$$\begin{aligned}\frac{1}{b-a} m_f\big((a,b)\big) &= \frac{1}{b-a}\big\|E\big((a,b)\big)f\big\|^2 \\ &\leq \sup_{a<\lambda<b,\,\mu>0}\langle f, \delta_{(\mu)}(H-\lambda)f\rangle = \sup_{a<\lambda<b,\,\mu>0}\frac{1}{\pi}\langle f, \Im R(\lambda+i\mu)f\rangle. \end{aligned}\tag{6.21}$$

Thus, if $J \subset \mathbb{R}$ is open and $|\frac{1}{\pi}\langle f, \Im R(\lambda + i\mu)f\rangle| \leq C(f) < \infty$ for all $\lambda \in J$ and $\mu > 0$, then m_f is absolutely continuous on J and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\|E_\lambda f\|^2 \leq C(f)$ on J (recall that $E_\lambda = E\big((-\infty,\lambda]\big)$) If this holds for each f in a dense subset of $\mathcal{H}$ then the spectrum of H in J is purely absolutely continuous.

Proposition 6.5.2. *Let $\mathcal{G}$ be a Banach space with $\mathcal{G} \subset \mathsf{D}(H^*)$ continuously and densely, and let $J \subset \mathbb{R}$ be open.*

i) If the GLAP holds for H locally on J, then H has purely absolutely continuous spectrum in J. If the strong GLAP holds for H in $\mathcal{G}$ and locally on J, then for each $\lambda_0 \in \mathbb{R}$ the function $\lambda \mapsto E_\lambda - E_{\lambda_0} \in \mathscr{B}(\mathcal{G},\mathcal{G}^*)$ is weak-continuously differentiable on J, and its derivative is equal to*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E_\lambda = \frac{1}{\pi}\Im R(\lambda + i0). \tag{6.22}$$

ii) Assume that $(\mathcal{G}^{*\circ})^* = \mathcal{G}$ *and that the GLAP holds in* $\mathcal{G}$ *locally on* J*. Let* $T : \mathsf{D}(H) \to \mathcal{H}$ *be a linear operator which is continuous when* $\mathsf{D}(H)$ *is equipped with the topology induced by* $\mathcal{G}^*$*, or in other terms let* $T \in \mathscr{B}(\mathcal{G}^{*\circ}, \mathcal{H})$*, then* T *is locally* H*-smooth on* J*.*

Proof. i) Let us first note that for any $f \in \mathsf{D}(H)^*$, the expression $\|E(\cdot)f\|^2$ is a well-defined positive Radon measure on $\mathbb{R}$, since for any bounded $J \in \mathcal{A}_B$ one has $E(J) \in \mathscr{B}\big(\mathsf{D}(H)^*, \mathsf{D}(H)\big)$. Note that this measure is usually unbounded if $f \in \mathsf{D}(H)^* \setminus \mathcal{H}$. As a consequence, (6.20) will hold in $\mathsf{D}(H)^*$, and by assumption (6.21) will hold for any $f \in \mathcal{G}$. It thus follows that m_f is absolutely continuous on J for any $f \in \mathcal{G}$. If the strong GLAP holds, then (6.22) is a direct consequence of (6.20).

ii) By assumption one has $T^* \in \mathscr{B}(\mathcal{H}, \mathcal{G})$. Since $\Im R(z)$ maps $\mathcal{G}$ into $\mathsf{D}(H) \subset \mathcal{G}^{*\circ}$ we get

$$\|T \Im R(z) T^*\| \leq \|T\|_{\mathcal{G}^* \to \mathcal{H}} \|\Im R(z)\|_{\mathcal{G} \to \mathcal{G}^*} \|T^*\|_{\mathcal{H} \to \mathcal{G}}$$

which means that T is locally H-smooth on J, by Proposition 6.4.2. □

Let us still mention that quite often, the limiting absorption principle is formally obtained by replacing $\Im R(\lambda + i\mu)$ by $R(\lambda + i\mu)$. However, since $R(\lambda + i\mu)$ does not belong to $\mathscr{B}\big(D(H)^*, \mathsf{D}(H)\big)$, the expression $\langle f, R(z) f\rangle$ is not *a priori* well-defined for f in the space $\mathcal{G}$ used before. One natural way to overcome this difficulty is to consider the space $\mathsf{D}\big(|H|^{1/2}\big)$, which is called *the form domain of* H. Then the following embeddings are continuous and dense

$$\mathsf{D}(H) \subset \mathsf{D}\big(|H|^{1/2}\big) \subset \mathcal{H} \subset \mathsf{D}\big(|H|^{1/2}\big)^* \subset \mathsf{D}(H)^*.$$

The main point in this construction is that $R(z) \in \mathscr{B}\Big(\mathsf{D}\big(|H|^{1/2}\big)^*, \mathsf{D}\big(|H|^{1/2}\big)\Big)$ for any $z \in \rho(H)$. So we can mimic the previous construction and consider $\mathcal{G} \subset \mathsf{D}\big(|H|^{1/2}\big)^*$ continuously and densely. Note that in the applications one often considers $\mathcal{G} \subset \mathcal{H} \subset \mathsf{D}\big(|H|^{1/2}\big)^*$. Then, it follows that that $\mathsf{D}\big(|H|^{1/2}\big) \subset \mathcal{G}^*$ continuously, and consequently

$$\mathscr{B}\Big(\mathsf{D}\big(|H|^{1/2}\big)^*, \mathsf{D}\big(|H|^{1/2}\big)\Big) \subset \mathscr{B}(\mathcal{G}, \mathcal{G}^{*\circ}) \subset \mathscr{B}(\mathcal{G}, \mathcal{G}^*).$$

In this context, the *limiting absorption principle* (LAP) or the strong LAP corresponds to the content of (6.5.1) with $\Im R(\lambda + i\mu)$ replaced by $R(\lambda + i\mu)$. Note that the LAP is usually a much stronger requirement than the GLAP since the the real part of $R(\lambda + i\mu)$ is a much more singular object (in the limit $\mu \searrow 0$) than its imaginary part.

Let us still reformulate the strong limiting absorption principle in different terms: Consider $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \pm\Im(z) > 0\}$ and observe that $\mathbb{C}_\pm \ni z \mapsto R(z) \in \mathscr{B}(\mathcal{G}, \mathcal{G}^*)$ is a holomorphic function. The strong LAP is equivalent to the fact that this function has a weak*-continuous extension to the set $\mathbb{C}_\pm \cup J$. The boundary values $R(\lambda \pm i0)$ of the resolvent on the real axis allow us to expression the derivative of the spectral measure on J as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E_\lambda = \frac{1}{2\pi i}\big[R(\lambda + i0) - R(\lambda - i0)\big]. \tag{6.23}$$

6.6 The method of differential inequalities

In this section we provide the proofs of Theorems 6.1.1 and 6.1.2. In fact, we shall mainly deal with a bounded operator S and show at the end how an unbounded operator H *with a spectral gap* can be treated in this setting. For an unbounded operator H without a spectral gap, we refer to [ABG, Sec. 7.5].

Our framework is the following: Let A be a self-adjoint (usually unbounded) operator in a Hilbert space $\mathcal{H}$ and let S be a bounded and self-adjoint operator in $\mathcal{H}$. We assume that S belongs to $C^{1,1}(A)$, as introduced in Section 6.2. Observe that this regularity condition means

$$
\begin{aligned}
&\int_{|t|\le 1} \big\|(\mathscr{U}_t - \mathbf{1})^2[S]\big\| \frac{\mathrm{d}t}{t^2} < \infty \\
\iff &\int_{|t|\le 1} \big\| \mathrm{e}^{2itA} S \,\mathrm{e}^{-2itA} - 2\,\mathrm{e}^{itA} S\,\mathrm{e}^{-itA} + S \big\| \frac{\mathrm{d}t}{t^2} < \infty \\
\iff &\int_{|t|\le 1} \big\| \mathrm{e}^{itA} S\,\mathrm{e}^{-itA} + \mathrm{e}^{-itA} S\,\mathrm{e}^{itA} - 2S \big\| \frac{\mathrm{d}t}{t^2} < \infty \\
\iff &\int_0^1 \big\| \mathrm{e}^{itA} S\,\mathrm{e}^{-itA} + \mathrm{e}^{-itA} S\,\mathrm{e}^{itA} - 2S \big\| \frac{\mathrm{d}t}{t^2} < \infty.
\end{aligned}
$$

Our first aim is to provide the proof the following theorem. In its statement, the space $\mathcal{G} := \big(\mathsf{D}(A), \mathcal{H}\big)_{1/2,1}$ appears, and we will explain its definition when necessary. The important information is that $\mathcal{G} \subset \mathcal{H}$ continuously and densely. We also recall that $\mu^A(S) = \{\lambda \in \mathbb{R} \mid \varrho_S^A(\lambda) > 0\}$.

Theorem 6.6.1. *Let S be a bounded and self-adjoint operator which belongs to $C^{1,1}(A)$. Then the holomorphic function $\mathbb{C}_\pm \ni z \mapsto (S-z)^{-1} \in \mathscr{B}(\mathcal{G}, \mathcal{G}^*)$ extends to a weak*-continuous function on $\mathbb{C}_\pm \cup \mu^A(S)$.*

As explained in the previous sections, such a statement means that a strong limiting absorption principle holds for S in $\mathcal{G}$ and locally on $\mu^A(S)$. Some consequences of this statement is that S has purely absolutely continuous spectrum on $\mu^A(S)$, that the derivative of its spectral measure can be expressed by the imaginary part of its resolvent on the real axis, as mentioned in (6.23), and that some locally S-smooth operators on $\mu^A(S)$ are automatically available.

The proof of Theorem 6.6.1 is divided into several lemmas. Note that since $S \in C^{1,1}(A) \subset C^1(A)$, then the commutator $B := [iS, A]$ is well-defined and belongs to $\mathscr{B}(\mathcal{H})$. Before starting with these lemmas, let us provide a heuristic explanation about the approach. Since the aim is to consider $\langle f, (S - \lambda \mp i\mu)^{-1} f\rangle$ and its limits when $\mu \searrow 0$, we shall consider a regularized version of such an expression, with an additional parameter ε, namely

$$\big\langle f_\varepsilon, \big(S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big)^{-1} f_\varepsilon \big\rangle. \tag{6.24}$$

Then, it is mainly a matter of playing with the ε-dependence of all these terms...

The following lemma is really technical, but it is of crucial importance since it is there that the assumption $\varrho_S^A(\lambda_0) > 0$ will play an essential role. For completeness we provide its proof. Note that the reason for getting an estimate of the form (6.25) will become clear only in the subsequent lemma.

Lemma 6.6.2. *Let $\{S_\varepsilon\}_{0<\varepsilon<1}$ and $\{B_\varepsilon\}_{0<\varepsilon<1}$ be two families of bounded self-adjoint operators satisfying $\|S_\varepsilon - S\| + \varepsilon\|B_\varepsilon\| \leq c\varepsilon$ for some constant c as well as the condition $\lim_{\varepsilon\to 0}\|B_\varepsilon - B\| = 0$. Let $\lambda_0 \in \mathbb{R}$ and $a \in \mathbb{R}$ such that $\varrho_S^A(\lambda_0) > a > 0$. Then there exist some strictly positive numbers δ, ε_0 and b such that for $|\lambda - \lambda_0| \leq \delta$, for $0 < \varepsilon \leq \varepsilon_0$ and for any $\mu \geq 0$ the following estimate holds for all $g \in \mathcal{H}$:*

$$a\|g\|^2 \leq \langle g, B_\varepsilon g\rangle + \frac{b}{\mu^2+\delta^2}\big\|\big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big]g\big\|^2. \tag{6.25}$$

Proof. Let us choose numbers $a < a_0 < a_1 < \varrho_S^A(\lambda_0)$ and $\delta > 0$ such that $a_1 E \leq EBE$ for $E = E^S\big((\lambda_0 - 2\delta, \lambda_0 + 2\delta)\big)$. Let us also choose $\varepsilon_1 > 0$ such that $\|B_\varepsilon - B\| \leq a_1 - a_0$ for any $0 < \varepsilon \leq \varepsilon_1$. This implies that $EB_\varepsilon E \geq EBE - (a_1 - a_0)E \geq a_0 E$. Let us set $E^\perp = \mathbf{1} - E$ and consider from now on λ and μ real with $|\lambda - \lambda_0| \leq \delta$ and $\mu \geq 0$. Observe then that $\|(S - \lambda \mp i\mu)^{-1}E^\perp\| \leq (\mu^2 + \delta^2)^{-1/2}$, and hence

$$\begin{aligned}
\|E^\perp g\|^2 &= \big\|(S-\lambda\mp i\mu)^{-1}E^\perp\big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu) + S - S_\varepsilon \pm i\varepsilon B_\varepsilon\big]g\big\|^2 \\
&\leq \frac{2}{\mu^2+\delta^2}\big\|\big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big]g\big\|^2 + \frac{2c^2\varepsilon^2}{\mu^2+\delta^2}\|g\|^2.
\end{aligned}$$

We then get for $\varepsilon \leq \varepsilon_1$ and for any $\nu > 0$:

$$\begin{aligned}
a_0\|g\|^2 &= a_0\langle g, Eg\rangle + a_0\|E^\perp g\|^2 \\
&\leq \langle g, EB_\varepsilon Eg\rangle + a_0\|E^\perp g\|^2 \\
&= \langle g, B_\varepsilon g\rangle - 2\Re\langle Eg, B_\varepsilon E^\perp g\rangle - \langle E^\perp g, B_\varepsilon E^\perp g\rangle + a_0\|E^\perp g\|^2 \\
&\leq \langle g, B_\varepsilon g\rangle + \nu\|g\|^2 + \nu^{-1}\|B_\varepsilon\|^2\|E^\perp g\|^2 + \|B_\varepsilon\|\,\|E^\perp g\|^2 + a_0\|E^\perp g\|^2 \\
&\leq \langle g, B_\varepsilon g\rangle + \nu\|g\|^2 + \big[\nu^{-1}\|B_\varepsilon\|^2 + \|B_\varepsilon\| + a_0\big]\frac{2c^2\varepsilon^2}{\mu^2+\delta^2}\|g\|^2 \\
&\quad + \big[\nu^{-1}\|B_\varepsilon\|^2 + \|B_\varepsilon\| + a_0\big]\frac{2}{\mu^2+\delta^2}\big\|\big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big]g\big\|^2.
\end{aligned}$$

Since we have $\|B_\varepsilon\| \leq c$ and since we may assume $c \geq 1$ and $\nu \leq 1$, it follows that

$$\Big[a_0 - \nu - \frac{2c^2\varepsilon^2(a_0+2c^2)}{\nu\delta^2}\Big]\|g\|^2 \leq \langle g, B_\varepsilon g\rangle + \frac{2a_0+4c^2}{\nu(\mu^2+\delta^2)}\big\|\big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big]g\big\|^2.$$

We can finally chose $\nu > 0$ and $\varepsilon_0 \in (0, \varepsilon_1)$ such that the term in the square brackets in the l.h.s. is bigger of equal to a for any $0 < \varepsilon \leq \varepsilon_0$. We then get (6.25) with $b = \nu^{-1}(2a_0 + 4c^2)$. □

Lemma 6.6.3. *Under the same assumptions as in the previous lemma, the operators* $S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)$ *are invertible in* $\mathscr{B}(\mathcal{H})$ *whenever* $|\lambda - \lambda_0| \leq \delta$, $0 < \varepsilon \leq \varepsilon_0$ *and* $\mu \geq 0$. *For any fixed* λ *and* μ *satisfying these conditions we set*

$$G_\varepsilon^\pm := \big[S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)\big]^{-1}$$

Then one has $(G_\varepsilon^\pm)^* = G_\varepsilon^\mp$ *and*

$$\|G_\varepsilon^\pm\| \leq \frac{1}{a\varepsilon + \mu}\Big[1 + b\varepsilon \frac{c\varepsilon + \|S\| + |\lambda + i\mu|}{\mu^2 + \delta^2}\Big]. \tag{6.26}$$

Moreover, for any $h \in \mathcal{H}$

$$\|G_\varepsilon^\pm h\| \leq \frac{1}{\sqrt{a\varepsilon}} |\Im\langle h, G_\varepsilon^+ h\rangle|^{1/2} + \frac{1}{\delta}\Big(\frac{b}{a}\Big)^{1/2} \|h\|. \tag{6.27}$$

Proof. Let us set

$$T_\varepsilon^\pm := S_\varepsilon - \lambda \mp i(\varepsilon B_\varepsilon + \mu)$$

and deduce from (6.25)

$$\begin{aligned}
(a\varepsilon + \mu)\|g\|^2 &\leq \langle g, (\varepsilon B_\varepsilon + \mu) g\rangle + \frac{b\varepsilon}{\mu^2 + \delta^2}\|T_\varepsilon^\pm g\|^2 \\
&= \mp \Im\langle g, T_\varepsilon^\pm g\rangle + \frac{b\varepsilon}{\mu^2 + \delta^2}\|T_\varepsilon^\pm g\|^2 \\
&\leq \|g\|\, \|T_\varepsilon^\pm g\| \Big\{1 + \frac{b\varepsilon}{\mu^2 + \delta^2}\|T_\varepsilon^\pm\|\Big\} \\
&\leq \|g\|\, \|T_\varepsilon^\pm g\| \Big\{1 + b\varepsilon \frac{\|S\| + c\varepsilon + |\lambda + i\mu|}{\mu^2 + \delta^2}\Big\}.
\end{aligned}$$

It follows from this equality and from the boundedness (and thus closeness) of $T_\varepsilon^\pm$ that these operators are injective and with closed range, see also [Amr, Lem. 3.1]. In addition, since $(T_\varepsilon^\pm)^* = T_\varepsilon^\mp$ and since $\mathsf{Ker}\big((T_\varepsilon^\pm)^*\big) = \mathsf{Ran}(T_\varepsilon^\pm)^\perp$, one infers that $\mathsf{Ran}(T_\varepsilon^\pm) = \mathcal{H}$. One then easily deduces all assertions of the lemma, except the estimate (6.27).

To prove (6.27), let us set $g = G_\varepsilon^\pm h$ in (6.25) and observe that

$$a\varepsilon \|G_\varepsilon^\pm h\|^2 \leq \langle G_\varepsilon^\pm h, (\varepsilon B_\varepsilon + \mu) G_\varepsilon^\pm h\rangle + \frac{b\varepsilon}{\mu^2 + \delta^2}\|h\|^2. \tag{6.28}$$

By taking the following identities into account,

$$\begin{aligned}
\langle G_\varepsilon^\pm h, (\varepsilon B_\varepsilon + \mu) G_\varepsilon^\pm h\rangle &= \pm (2i)^{-1} \langle h, G_\varepsilon^\mp (T_\varepsilon^\mp - T_\varepsilon^\pm) G_\varepsilon^\pm h\rangle \\
&= \pm (2i)^{-1} \langle h, (G_\varepsilon^\pm - G_\varepsilon^\mp) h\rangle \\
&= \Im\langle h, G_\varepsilon^+ h\rangle,
\end{aligned}$$

one directly obtains (6.27) from (6.28). $\square$

We keep the assumptions and the notations of the previous two lemmas, and set simply G_ε and T_ε for G_ε^+ and T_ε^+. The derivative with respect to the variable ε will be denoted by $'$, *i.e.* $G_\varepsilon' = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}G_\varepsilon$.

Lemma 6.6.4. *Assume that the maps $\varepsilon \mapsto S_\varepsilon$ and $\varepsilon \mapsto B_\varepsilon$ are C^1 in norm, and that for any fixed ε the operators B_ε and S_ε belong to $C^1(A)$. Then the map $\varepsilon \mapsto G_\varepsilon$ is C^1 in norm, and also $G_\varepsilon \in C^1(A)$ for any fixed ε.*

Proof. The differentiability of G_ε with respect to ε follows easily from the equality

$$G_{\varepsilon'} - G_\varepsilon = G_{\varepsilon'}(T_\varepsilon - T_{\varepsilon'})G_\varepsilon = G_{\varepsilon'}\big(S_\varepsilon - S_{\varepsilon'} - i(\varepsilon B_\varepsilon - \varepsilon' B_{\varepsilon'})\big)G_\varepsilon.$$

We thus get that $G_\varepsilon' = -G_\varepsilon T_\varepsilon' G_\varepsilon$. For the regularity of G_ε with respect to A, this follows from the statement (iii) of Proposition 6.2.4. In addition, one can also infer that $[A, G_\varepsilon] = -G_\varepsilon[A, T_\varepsilon]G_\varepsilon$, see [ABG, Prop. 6.1.6] for the details. Note that this equality can also be obtained with the operator A_τ, as already used in (6.11). □

Note that from the previous statement and from its proof, one deduces the following equality

$$G_\varepsilon' + [A, G_\varepsilon] = iG_\varepsilon\big\{B_\varepsilon - [iS_\varepsilon, A] + \varepsilon\big(i\varepsilon^{-1}S_\varepsilon' + B_\varepsilon' + [A, B_\varepsilon]\big)\big\}G_\varepsilon. \tag{6.29}$$

In the next lemma we provide a precise formulation of what had been mentioned in (6.24).

Lemma 6.6.5. *Let us keep the assumptions of the previous three lemmas, and let $\{f_\varepsilon\}_{0<\varepsilon<1}$ be a bounded family of elements of $\mathcal{H}$ such that $\varepsilon \mapsto f_\varepsilon$ is strongly C^1 and such that $f_\varepsilon \in \mathsf{D}(A)$ for any fixed ε. Set*

$$F_\varepsilon := \langle f_\varepsilon, G_\varepsilon f_\varepsilon\rangle.$$

Then the map $\varepsilon \mapsto F_\varepsilon$ is of class C^1 and its derivatives satisfies

$$F_\varepsilon' = \langle f_\varepsilon, (G_\varepsilon' + [A, G_\varepsilon])f_\varepsilon\rangle + \langle G_\varepsilon^* f_\varepsilon, f_\varepsilon' + Af_\varepsilon\rangle + \langle f_\varepsilon' - Af_\varepsilon, G_\varepsilon f_\varepsilon\rangle. \tag{6.30}$$

In addition, if we set

$$\ell(\varepsilon) := \|f_\varepsilon'\| + \|Af_\varepsilon\|, \qquad q(\varepsilon) := \|\varepsilon^{-1}(B_\varepsilon - [iS_\varepsilon, A]) + i\varepsilon^{-1}S_\varepsilon' + B_\varepsilon' + [A, B_\varepsilon]\|, \tag{6.31}$$

then F_ε satisfies the differential inequality

$$\frac{1}{2}|F_\varepsilon'| \leq \omega\|f_\varepsilon\|\big[\ell(\varepsilon) + \omega\varepsilon q(\varepsilon)\|f_\varepsilon\|\big] + \frac{\ell(\varepsilon)}{\sqrt{a\varepsilon}}|F_\varepsilon|^{1/2} + \frac{q(\varepsilon)}{a}|F_\varepsilon|, \tag{6.32}$$

with $\omega = a^{-1/2}b^{1/2}\delta^{-1}$ and $0 < \varepsilon \leq \varepsilon_0$.

Proof. The equality (6.30) is obvious, since the commutator can be opened on $\mathsf{D}(A)$. By using (6.29) it can then be rewritten as

$$\begin{aligned} F_\varepsilon' =& i\big\langle G_\varepsilon^* f_\varepsilon, \big\{B_\varepsilon - [iS_\varepsilon, A] + \varepsilon(i\varepsilon^{-1}S_\varepsilon' + B_\varepsilon' + [A, B_\varepsilon])\big\}G_\varepsilon f_\varepsilon\big\rangle \\ &+ \langle G_\varepsilon^* f_\varepsilon, f_\varepsilon' + Af_\varepsilon\rangle + \langle f_\varepsilon' - Af_\varepsilon, G_\varepsilon f_\varepsilon\rangle. \end{aligned} \tag{6.33}$$

As a consequence one infers that

$$|F_\varepsilon'| \le \varepsilon q(\varepsilon)\|G_\varepsilon f_\varepsilon\|\,\|G_\varepsilon^* f_\varepsilon\| + \ell(\varepsilon)\big(\|G_\varepsilon f_\varepsilon\| + \|G_\varepsilon^* f_\varepsilon\|\big). \tag{6.34}$$

In addition it follows from (6.27) that

$$\|G_\varepsilon^\pm f_\varepsilon\| \le \frac{1}{\sqrt{a\varepsilon}}|F_\varepsilon|^{1/2} + \omega\|f_\varepsilon\|.$$

By inserting these inequalities in (6.34) and by using the inequality $(p+q)^2 \le 2p^2+2q^2$ for any $p, q \ge 0$ one directly obtains (6.32). □

The differential inequality (6.32), from which the method takes its name, is quite remarkable in that the spectral variable $\lambda + i\mu$ does not appear explicitly in the coefficients. In fact, the only conditions on these parameters are $|\lambda - \lambda_0| \le \delta$ and $\mu \ge 0$.

Let us still rewrite (6.32) in the simple form

$$|F_\varepsilon'| \le \eta(\varepsilon) + \varphi(\varepsilon)|F_\varepsilon|^{1/2} + \psi(\varepsilon)|F_\varepsilon|.$$

By using the trivial inequality $|F_s| \le |F_{\varepsilon_0}| + \int_s^{\varepsilon_0} |F_\tau'|\mathrm{d}\tau$, we then obtain for $0 < \varepsilon < s < \varepsilon_0$

$$|F_s| \le |F_{\varepsilon_0}| + \int_s^{\varepsilon_0} \eta(\tau)\mathrm{d}\tau + \int_s^{\varepsilon_0} \big[\varphi(\tau)|F_\tau|^{1/2} + \psi(\tau)|F_\tau|\big]\mathrm{d}\tau. \tag{6.35}$$

We shall now apply an extended version of the Gronwall lemma to this differential inequality. More precisely, let us first state such a result, and refer to [ABG, Lem. 7.A.1] for its proof.

Lemma 6.6.6 (Gronwall lemma)**.** *Let $(a, b) \subset \mathbb{R}$ and let f, φ, ψ be non-negative real functions on (a, b) with f bounded, and $\varphi, \psi \in L^1\big((a, b)\big)$. Assume that, for some constants $\omega \ge 0$ and $\theta \in [0, 1)$ and for all $\lambda \in (a, b)$ one has*

$$f(\lambda) \le \omega + \int_\lambda^b \big[\varphi(\tau)f(\tau)^\theta + \psi(\tau)f(\tau)\big]\mathrm{d}\tau.$$

Then one has for each $\lambda \in (a, b)$

$$f(\lambda) \le \Big[\omega^{1-\theta} + (1-\theta)\int_\lambda^b \varphi(\tau)\,\mathrm{e}^{(\theta-1)\int_\tau^b \psi(s)\mathrm{d}s}\,\mathrm{d}\tau\Big]^{1/(1-\theta)}\,\mathrm{e}^{\int_\lambda^b \psi(\tau)\mathrm{d}\tau}.$$

Thus, by applying this result for $\theta = 1/2$ to (6.35) one gets that

$$|F_\varepsilon| \leq \Big[\Big(|F_{\varepsilon_0}| + \int_\varepsilon^{\varepsilon_0} \eta(\tau)\mathrm{d}\tau\Big)^{1/2} + \frac{1}{2}\int_\varepsilon^{\varepsilon_0} \varphi(\tau)\,\mathrm{e}^{-\frac{1}{2}\int_\tau^{\varepsilon_0}\psi(s)\mathrm{d}s}\,\mathrm{d}\tau\Big]^2\,\mathrm{e}^{\int_\varepsilon^{\varepsilon_0}\psi(\tau)\mathrm{d}\tau}$$

for all $0 < \varepsilon < \varepsilon_0$. We can then deduce the simpler inequality

$$|F_\varepsilon| \leq 2\Big[|F_{\varepsilon_0}| + \int_\varepsilon^{\varepsilon_0} \eta(\tau)\mathrm{d}\tau + \Big(\int_\varepsilon^{\varepsilon_0} \varphi(\tau)\mathrm{d}\tau\Big)^2\Big]\,\mathrm{e}^{\int_\varepsilon^{\varepsilon_0}\psi(\tau)\mathrm{d}\tau}.$$

Our final purpose is to get a bound on $|F_\varepsilon| < const. < \infty$ independent of $z = \lambda + i\mu$ as $\varepsilon \to 0$. From the above estimate we see that this is satisfied if

$$\int_0^{\varepsilon_0} [\eta(\tau) + \varphi(\tau) + \psi(\tau)]\mathrm{d}\tau < \infty.$$

By coming back to the explicit formula for these functions, it corresponds to the condition

$$\int_0^{\varepsilon_0} \Big[\ell(\varepsilon)\|f_\varepsilon\| + \varepsilon q(\varepsilon)\|f_\varepsilon\|^2 + \frac{\ell(\varepsilon)}{\sqrt{\varepsilon}} + q(\varepsilon)\Big]\mathrm{d}\varepsilon < \infty.$$

In fact, it is easily observed (see also page 304 of [ABG]) that this condition is satisfied if the following assumption holds:

$$\int_0^1 [\varepsilon^{-1/2}\ell(\varepsilon) + q(\varepsilon)]\mathrm{d}\varepsilon < \infty.$$

By looking back at the definitions of ℓ and q in (6.31) we observe that the condition on ℓ corresponds to a condition on the family of elements $\{f_\varepsilon\}$ while the condition on q corresponds to conditions on the families $\{S_\varepsilon\}$ and $\{B_\varepsilon\}$. For the condition on q let us just mention that a suitable choice for S_ε is given by

$$S_\varepsilon := \int_{-\infty}^{\infty} \mathrm{e}^{-i\varepsilon\tau A}\,S\,\mathrm{e}^{i\varepsilon\tau A}\,\mathrm{e}^{-\tau^2/4}\,\frac{\mathrm{d}\tau}{(4\pi)^{1/2}}.$$

Then by setting $B_\varepsilon := [iS_\varepsilon, A]$ and by assuming that $S \in C^{1,1}(A)$ it is shown in [ABG, Lem. 7.3.6] that all assumptions on the families $\{S_\varepsilon\}$ and $\{B_\varepsilon\}$ are satisfied. Note that the proof of this statement is rather technical and that we shall not comment on it.

For the condition involving ℓ, let us consider $f \in \mathcal{H}$ and set for any $\varepsilon > 0$

$$f_\varepsilon := (\mathbf{1} + i\varepsilon A)^{-1} f.$$

Then one has $f_\varepsilon \in \mathsf{D}(A)$, $f_\varepsilon' = -i(\mathbf{1} + i\varepsilon A)^{-1} A f_\varepsilon$, $\|f_\varepsilon\| \leq \|f\|$, $\|f_\varepsilon - f\| \to 0$ as $\varepsilon \to 0$, and $\ell(\varepsilon) \leq 2\|Af_\varepsilon\|$. Then the condition $\int_0^1 \varepsilon^{-1/2}\ell(\varepsilon)\mathrm{d}\varepsilon < \infty$ holds if

$$\int_0^1 \varepsilon^{1/2}\big\|A(\mathbf{1} + i\varepsilon A)^{-1} f\big\|\frac{\mathrm{d}\varepsilon}{\varepsilon} < \infty. \tag{6.36}$$

Such a condition corresponds to a regularity condition of f with respect to A. In fact, many Banach spaces of elements of $\mathcal{H}$ having a certain regularity with respect A can be defined, and Chapter 2 of [ABG] is entirely devoted to that question. Here, the elements of $\mathcal{H}$ satisfying condition (6.36) are precisely those belonging to the space $\big(\mathsf{D}(A), \mathcal{H}\big)_{1/2,1}$, as shown in [ABG, Prop. 2.7.2]. Note that this space is an interpolation space between $\mathsf{D}(A)$ and $\mathcal{H}$ and contains the space $\mathsf{D}(\langle A\rangle^{1/2+\epsilon})$ for any $\epsilon > 0$.

By summing up, for any $f \in \mathcal{G} := \big(\mathsf{D}(A), \mathcal{H}\big)_{1/2,1}$ one has $\int_0^1 \varepsilon^{-1/2}\ell(\varepsilon)\mathrm{d}\varepsilon \leq c_1\|f\|_{\mathcal{G}}$ for some $c_1 < \infty$ independent of $f \in \mathcal{G}$. By looking at the explicit form of the functions η and φ one also obtains that there exists $c_2 < \infty$ such that $\int_0^1 \eta(\tau)\mathrm{d}\tau \leq c_2\|f\|_{\mathcal{G}}^2$ and $\int_0^1 \varphi(\tau)\mathrm{d}\tau \leq c_2\|f\|_{\mathcal{G}}$. One then infers that $|F_\varepsilon| \leq c\|f\|_{\mathcal{G}}^2$ for any $0 < \varepsilon \leq \varepsilon_0$, $|\lambda - \lambda_0| \leq \delta$ and $\mu \geq 0$ with a constant c independent of $f \in \mathcal{G}$, ε, λ and μ. The proof of Theorem 6.6.1 can now be provided:

Proof of Theorem 6.6.1. By all the previous arguments, there exists an integrable function $\kappa : (0, \varepsilon_0) \to \mathbb{R}$ such that $|F_\varepsilon'| \leq \kappa(\varepsilon)$ for all ε, λ, μ as above. Now, fix $\mu > 0$. Since $S_\varepsilon - \lambda - i(\varepsilon B_\varepsilon + \mu)$ converges to $S - \lambda - i\mu \equiv S - z$ in norm as $\varepsilon \to 0$ we shall have $G_\varepsilon \to (S - z)^{-1}$ in norm too, and

$$\langle f, (S - z)^{-1} f\rangle = \lim_{\varepsilon \to 0}\langle f_\varepsilon, G_\varepsilon f_\varepsilon\rangle = \langle f_{\varepsilon_0}, G_{\varepsilon_0}(z) f_{\varepsilon_0}\rangle - \int_0^{\varepsilon_0} F_\varepsilon'(z)\mathrm{d}\varepsilon. \tag{6.37}$$

Note that we have explicitly indicated the dependence on $z = \lambda + i\mu$ of G_{ε_0} and F_ε'. Let us set $\Omega := \{\lambda + i\mu \mid |\lambda - \lambda_0| < \delta, \mu \geq 0\}$. It follows from (6.26) that $\|G_{\varepsilon_0}(z)\| \leq$ *const.* $< \infty$ independently of $z \in \Omega$. For each $\varepsilon > 0$ the continuity of $z \in \Omega$ of $F_\varepsilon'(z)$ follows from (6.33). By the dominated convergence theorem, with the fact that $|F_\varepsilon'| \leq \kappa(\varepsilon)$, the equation (6.37) gives the existence of a continuous extension of the function $\langle f, (S - z)^{-1} f\rangle$ from the domain $\{z \in \Omega \mid \Im(z) = \mu > 0\}$ to all Ω. The polarization principle shows that this holds for $\langle f, (S - z)^{-1} g\rangle$ for any $f, g \in \mathcal{G}$. $\square$

Let us finally show how the two theorems stated at the beginning of the chapter follow from the various results obtained subsequently. First of all we provide a proof of Theorem 6.1.2.

Proof of Theorem 6.1.2. Let $\lambda_0 \in \mathbb{R} \setminus \sigma(H)$ and set $S := (H - \lambda_0)^{-1}$. Then S is a bounded self-adjoint operator, and the resolvents of S and H are related by the identity

$$(H - z)^{-1} = (\lambda_0 - z)^{-1}[S - (\lambda_0 - z)^{-1}]^{-1} S, \qquad \Im(z) \neq 0. \tag{6.38}$$

Let $J \subset \mu^A(H)$ be a compact set with $\lambda_0 \notin J$ and set $\tilde{J} := \{(\lambda_0 - \lambda)^{-1} \mid \lambda \in J\}$. Note that there is no restriction on the generality if we assume that λ does not belong to a neighbourhood of λ_0, since $(H - z)^{-1}$ is holomorphic in such a neighbourhood. Then $S \in C^{1,1}(A)$ and $\tilde{J}$ is a compact subset of $\mu^A(S)$, see Proposition 7.2.5 of [ABG]. In addition, Theorem 6.6.1 says that the map $\zeta \mapsto (S - \zeta)^{-1} \in \mathscr{B}(\mathcal{G}, \mathcal{G}^*)$ extends to a weak*-continuous function on $\mathbb{C}_\pm \cup \tilde{J}$. Since $z \mapsto \zeta = (\lambda_0 - z)^{-1}$ is a homeomorphism of $\mathbb{C}_\pm \cup J$ onto $\mathbb{C}_\pm \sup \tilde{J}$, we see that $z \mapsto [S - (\lambda_0 - z)^{-1}]^{-1} \in \mathscr{B}(\mathcal{G}, \mathcal{G}^*)$ extends to a weak*-continuous function on $\mathbb{C}_\pm \cup J$. The result of the theorem now follows from the identity (6.38) and the fact that $S\mathcal{G} \subset \mathcal{G}$, as a consequence of [ABG, Thm. 5.3.3].

The second part of the statement is a direct consequence of what has been presented in Section 6.4. Note in particular that in the Definition 6.4.1 of a locally H-smooth operator, one could have considered $T : \mathsf{D}(H) \to \mathcal{K}$ with $\mathcal{K}$ an arbitrary Hilbert space. It is in this generality that the statement of Theorem 6.1.2 is provided, and this slight extension can easily be taken into account. □

In the same vein one has:

Proof of Theorem 6.1.1. The first assertion about the finiteness of the set of eigenvalues is a direct consequence of Corollary 6.3.7. For the second statement, observe that Theorem 6.3.8 implies the inclusion $J \setminus \sigma_p(H) \subset \mu^A(H)$, and then use the consequences of the limiting absorption principle, as presented in Section 6.5. □