Lesson 12
Application of Integrals (1)

12A
• Areas of plane regions
Area of a Plane Region

Area of the strip \[ \Delta S = f(x) \, dx \] where \( f(x) > 0 \)

Total area over \([a, b]\) \[ S \approx \sum f(x) \Delta x \quad \rightarrow \quad S = \int_{a}^{b} f(x) \, dx \]
Area Between Two Graphs

When \( f(x) \geq g(x) \) in \([a, b]\), area \( S \) is

\[
S = \int_a^b \{f(x) - g(x)\} \, dx
\]

[Proof]

Consider \( y = f(x) + k \) and \( y = g(x) + k \)

\[
S = S_1 - S_2 = \int_a^b \{f(x) + k\} \, dx - \int_a^b \{g(x) + k\} \, dx = \int_a^b \{f(x) - g(x)\} \, dx
\]
Case of a Negative Function

When \( f(x) < 0 \) in the domain \([a, b]\)

\[
S = \int_a^b \{0 - f(x)\} \, dx = -\int_a^b f(x) \, dx
\]

Ahh! That’s so easy!
Example 12-1  Find the area between the graph of \( y = x^2 - x - 2 \) and the \( x \)-axis, from \( x = -2 \) to \( x = 3 \).

**Ans.**

\[
y = x^2 - x - 2 = (x - 2)(x + 1) = 0
\]

Let the areas be \( S_1, \ S_2 \) and \( S_3 \)

\[
S_1 = \int_{-2}^{-1} (x^2 - x - 2)dx = \left[ \frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-2}^{-1} = 1.833
\]

\[
S_2 = -\int_{-1}^{2} (x^2 - x - 2)dx = 4.5
\]

\[
S_3 = -\int_{2}^{3} (x^2 - x - 2)dx = 1.833
\]

Therefore, \( S = S_1 + S_2 + S_3 \approx 1.833 + 4.5 + 1.833 = 8.166 \)
Example 12-2  Determine the area of the region enclosed by the functions
and $y = \sqrt{x}$, $y = x^2$

Ans.

The area is given by the blue region in the figure.
Cross points:

$$\sqrt{x} = x^2$$

$$x = x^4$$, \quad \therefore \quad x(x - 1)(x^2 + x + 1) = 0$$

$$(0, 0), \quad (1, 1)$$

$$S = \int_{0}^{1} (\sqrt{x} - x^2) \, dx = \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_{0}^{1} = \frac{1}{3}$$
Area Between the graph and the y-Axis

The area between the graph $x = g(y)$ and the $y$-axis

$$S = \int_{c}^{d} g(y)\,dy$$
Ex.12-1  Find the area of the region enclosed by $y = \sin 2x$ and $y = \cos x$ in the domain $0 \leq x \leq \frac{\pi}{2}$.

Ans.

Pause the video and solve by yourself.
**Ex.12-1** Find the area of the region enclosed by \( y = \sin 2x \) and \( y = \cos x \) in the domain \( 0 \leq x \leq \frac{\pi}{2} \).

**Ans.** We put \( \sin 2x = \cos x \)

\[ \therefore \cos x(2\sin x - 1) = 0, \]

\[ \therefore \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2} \]

\[ \therefore x = \frac{\pi}{2}, \frac{\pi}{6} \]

\[ S = \int_{0}^{\frac{\pi}{6}} (\cos x - \sin 2x) \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x) \, dx \]

\[ = \left[ \sin x - \frac{1}{2} \cos 2x \right]_{0}^{\frac{\pi}{6}} + \left[ -\frac{1}{2} \cos 2x - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2} \]
Lesson 12
Application of Integrals (1)

12B
• Volumes of Solids
Volumes of Solids

Volume of the slab \( \Delta V \approx S(x) \Delta x \)

Total volume of the solid between \( x=a \) and \( x=b \)

\[ V \approx \sum S(x) \Delta x \quad \rightarrow \quad V = \int_a^b S(x) \, dx \]
Example

[Example 12-2]
Find the volume of a cone with bottom radius $r$ and height $h$.

Ans.

Set the $x$-axis as shown in the figure.

Area of the bottom $S(h) = \pi r^2$

From $S(x) : S(h) = x^2 : h^2$, we have

$S(x) = \frac{\pi r^2}{h^2} x^2$

Therefore

$$V = \int_0^h \left( \frac{\pi r^2}{h^2} x^2 \right) dx = \frac{\pi r^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h = \frac{\pi}{3} r^2 h$$
Ex.12-2  Find the volume of the pyramid having a horizontal square cross section. The bottom side length is $a$ and the height is $h$.

Ans.

Pause the video and solve by yourself.
Ex.12-2 Find the volume of the pyramid having a horizontal square cross section. The bottom side length is $a$ and the height is $h$.

Ans. We consider the z-axis vertically.

The horizontal cross section at $z$:

$$ S(z) : a^2 = (h - z)^2 : h^2 $$

$$ \therefore S(z) = \frac{a^2 (h - z)^2}{h^2} $$

$$ V = \int_0^h \frac{a^2 (h - z)^2}{h^2} \, dz = \frac{a^2}{h^2} \int_0^h (h^2 - 2hz + z^2) \, dz $$

$$ = \frac{a^2}{h^2} \left[ h^2 z - hz^2 + \frac{1}{3} z^3 \right]_0^h = \frac{1}{3} a^2 h $$
Volumes of Solids of Revolution

When the solid is generated by revolving a region about the x-axis

\[ V = \int_a^b S(x) \, dx = \int_a^b \pi r^2 \, dx = \int_a^b \pi f(x)^2 \, dx \]
[Examples 12-3]
Find the volume of a sphere with radius \( r \).

Ans.

The upper half of the circle.

\[ y = \sqrt{r^2 - x^2} \]

By rotating this blue area, we have a circle.

Volume of a circle

\[
V = \int_{-r}^{r} \pi y^2 \, dx = \int_{-r}^{r} \pi (r^2 - x^2) \, dx
\]

\[ = \pi \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4}{3} \pi r^3 \]

That makes sense!
Ex.12-3  Find the volume of the solid made by rotating the region surrounded by $f(x) = \sqrt{x}$ and $y=x$.

Ans.

Pause the video and solve by yourself.
Ex.12-3 Find the volume of the solid made by rotating the region surrounded by \( y = \sqrt{x} \) and \( y = x \).

Ans.

This volume can be obtained by subtracting B from A, where

\[
V_A = \pi \int_0^1 \left( \sqrt{x} \right)^2 \, dx = \pi \left[ \frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} \pi
\]

\[
V_B = \pi \int_0^1 \left( x \right)^2 \, dx = \pi \left[ \frac{1}{3} x^3 \right]_0^1 = \frac{1}{3} \pi
\]

\[
V = V_B - V_A = \frac{1}{3} \pi - \frac{1}{2} \pi = \frac{1}{6} \pi
\]