Lesson 8
Approximation of a Function

8A
• Linear Approximation
Definition of the Derivative

\[ \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a) \]

When \( |h| \) is small

\[ \frac{f(a + h) - f(a)}{h} \approx f'(a) \]

Rearrange

\[ f(a + h) \approx f(a) + f'(a)h \]

If we put \( a + h = x \)

\[ f(x) \approx f(a) + f'(a)(x - a) \]
Example 8-1 Derive the approximate expression of $f(x) = \sqrt{x}$ in the neighborhood of $a = 1$.

**Ans.**

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

At $a = 1$

$$f(1) = \sqrt{1} = 1 \text{ and } f'(1) = \frac{1}{2}$$

Therefore

$$\sqrt{x} \approx f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$$

For example

$$x = 1.1 \text{ Calculator } \sqrt{1.1} = 1.0488...$$

This approximation $\sqrt{1.1} = \frac{1}{2} \times 1.1 + \frac{1}{2} = 1.05$  \{ Error 1.1\% \}
Example 8-2 (1) When \( h \) is small, find the linear approximation of \( \sin(a + h) \).

(2) Find the approximate value of \( \sin 31^\circ \).

**Ans.**

(1) \( f(x) = \sin x \implies f'(x) = \cos x \)

\[
\sin(a + h) = f(a + h) \approx f(a) + f'(a)h = \sin a + h \cos a
\]

(2)

\[
31^\circ = \frac{\pi}{6} + \frac{\pi}{180}
\]

We put \( h = \frac{\pi}{180} \)

\[
\sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \sin \frac{\pi}{6} + \frac{\pi}{180} \cos \frac{\pi}{6} = \frac{1}{2} + \frac{\pi}{180} \cdot \frac{\sqrt{3}}{2}
\]

Using \( \pi = 3.142, \sqrt{3} = 1.732 \)

We have \( \sin 31^\circ \approx \frac{1}{2} + \frac{3.142}{180} \cdot \frac{1.732}{2} \approx 0.5151 \)
Case that \(|x|\) is small

\[ a + h = x \]
\[ f(x) \approx f(a) + f'(a)(x - a) \]

When \(x\) is small \quad \text{We put} \quad a = 0.

\[ f(x) \approx f(0) + f'(0)x \]

**Examples 8-3** When \(|x|\) is small enough, derive the approximate expression of \(\sqrt{1 + x}\)

\[ \sqrt{1 + x}' = \frac{1}{2\sqrt{1 + x}} \]

Therefore

\[ f(x) \approx f(0) + f'(0)x = 1 + \frac{1}{2}x \]
Ex.8-1 Find the approximate value of $\cos 29^\circ$ in the following two ways.

(1) Use the formula $f(x) = f(a + h) \approx f(a) + f'(a)h$

(2) Use the formula $f(x) \approx f(0) + f'(0)x$

Ans.

Pause the video and solve the problem by yourself.
Ex.8-1 Find the approximate value of \( \cos 29^\circ \) in the following two ways.

(1) Use the formula \( f(x) = f(a + h) \approx f(a) + f'(a)h \)

(2) Use the formula \( f(x) \approx f(0) + f'(0)x \)

Ans.

(1) We put \( f(x) = \cos x \)

Then \( f'(x) = -\sin x \)

\[
\cos 29^\circ = \cos(\frac{\pi}{6} - \frac{\pi}{180}) \approx \cos(\frac{\pi}{6}) - \frac{\pi}{180} \sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{180} \cdot \frac{1}{2}
\]

\[
\approx \frac{1.732}{2} - \frac{3.142}{180} \cdot \frac{1}{2} = 0.875
\]

(2) We put \( f(x) = \cos\left(\frac{\pi}{6} + x\right) \)

Then \( f'(x) = -\sin\left(\frac{\pi}{6} + x\right) \)

\[
\cos 29^\circ = f(-\frac{\pi}{180}) \approx \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right)
\]

\[
= \frac{\sqrt{3}}{2} + \frac{\pi}{180} \cdot \frac{1}{2} \approx \frac{1.732}{2} + \frac{3.142}{180} \cdot \frac{1}{2} = 0.875
\]
Lesson 8
Application of a Function

8B

• Tayler Expansion
Suppose that \( f(x) \) is expanded in power series

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + ...
\]

By differentiating term by term, we have

\[
f'(x) = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + ...
\]

\[
f''(x) = 2a_2 + 2 \cdot 3a_3(x - c) + 3 \cdot 4a_4(x - c)^2 + ...
\]

......

Setting \( x = c \), we have

\[
f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad ...... \quad f^{(k)}(c) = k!a_k, \quad ......
\]

Namely, the coefficients are given by

\[
a_k = \frac{f^{(k)}(c)}{k!}
\]
Taylor series

For a function $f(x)$ that has continuous derivatives in the neighborhood of $c$, the Taylor series expansion of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

$$= f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \ldots$$

For $c = 0$

Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$
**Example 8-3** Derive the approximate expression for \( \sin x \) near \( x = 0 \) up to the seventh order.

**Ans.** We put \( f(x) = \sin x \)

Then \( f'(x) = \cos x \), \( f''(x) = -\sin x \)

In general \( f^{(2n)}(x) = (-1)^n \sin x \), \( f^{(2n+1)}(x) = (-1)^n \cos x \)

For \( x = 0 \) : \( f^{(2n)}(0) = 0 \), \( f^{(2n+1)}(0) = (-1)^n \)

Using these values

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
\]
Ex.8-2 Find Maclaurin series for \( f(x) = e^x \)

Ans.

Pause the video and solve the problem by yourself.
Ex.8-2 Find Maclaurin series for $f(x) = e^x$

Ans.

The $n$-th derivative is $f^{(n)}(x) = e^x$ for all $n$.

Thus, these values for $x = 0$ are

$$f(0) = f'(0) = f''(0) = \ldots = e^0 = 1$$

The coefficients are

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$$

Therefore

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$