Lesson 5
Derivatives of Logarithmic Functions and Exponential Functions

5A
• Derivative of logarithmic functions
Review of the Logarithmic Function

Exponential function

\[ y = a^x \quad (a > 0, \ a \neq 1) \]

Logarithmic function

\[ \log_a y = x \]

We replace the notation

\[ x = a^y \quad \leftrightarrow \quad y = \log_a x \]
Derivative of the Logarithmic Function

From the definition

\[
(\log_a x)' = \lim_{h \to 0} \frac{\log_a (x + h) - \log_a x}{h}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \log_a \left(1 + \frac{h}{x}\right) = \lim_{h \to 0} \left\{ \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right) \right\}
\]

We put \( \frac{h}{x} = k \). When \( h \to 0, \ k \to 0 \).

\[
(\log_a x)' = \lim_{k \to 0} \left\{ \frac{1}{x} \cdot \frac{1}{k} \log_a (1 + k) \right\}
\]

\[
= \frac{1}{x} \lim_{k \to 0} \log_a \left(1 + k\right)^{\frac{1}{k}} = \frac{1}{x} \log_a \left[ \lim_{k \to 0} \left(1 + k\right)^{\frac{1}{k}} \right]
\]
Napier’s Constant

### Trial

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(1 + k)^{1/k}$</th>
<th>$k$</th>
<th>$(1 + k)^{1/k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.59374·······</td>
<td>−0.1</td>
<td>2.86797·······</td>
</tr>
<tr>
<td>0.01</td>
<td>2.70481·······</td>
<td>−0.01</td>
<td>2.73199·······</td>
</tr>
<tr>
<td>0.001</td>
<td>2.71692·······</td>
<td>−0.001</td>
<td>2.71964·······</td>
</tr>
<tr>
<td>0.0001</td>
<td>2.71814·······</td>
<td>−0.0001</td>
<td>2.71841·······</td>
</tr>
<tr>
<td>0.00001</td>
<td>2.71826·······</td>
<td>−0.00001</td>
<td>2.71829·······</td>
</tr>
</tbody>
</table>

We expect that $(1 + k)^{1/k}$ approaches one value as $k \to 0$.

**Napier’s Constant**

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828459045\cdots$$

**Important Mathematical Constants**

“π=3.1415…” was known 4000 years ago

“e=2.7182…” was found in 17th century
Then

\[(\log_a x)' = \frac{1}{x \log_a e} = \frac{1}{x \log_e a}\]

If the base is \(e\), we have

\[(\log_e x)' = \frac{1}{x \log_e e} = \frac{1}{x}\]

Natural logarithm is the logarithm to the base \(e\).

Notation:

\[\log_e x \rightarrow \ln x\]

Summary

\[
\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}
\]
[Example 5.1] Find the derivative of the following functions.

(1)  \( y = \ln 2x \),  
(2)  \( y = \log_2 (3x + 2) \),  
(3)  \( y = x \ln 3x \)

Ans.

(1)  \( y' = \frac{1}{2x} \cdot (2x)' = \frac{2}{2x} = \frac{1}{x} \)

(2)  \( y' = \frac{1}{(3x + 2) \ln 2} \cdot (3x + 2)' = \frac{3}{(3x + 2) \ln 2} \)

(3)  \( y' = x' \ln 3x + x(\ln 3x)' = \ln 3x + x \cdot \frac{3}{3x} = \ln 3x + 1 \)
[ Example 5.2] Calculate the money which you can receive one year later using various compound systems. The principal is 10000 yen. (1) Annual interest is 100%. (2) Half a year interest is 50%, (3) Monthly interest is 100/12%, (4) Daily interest is 100/360%.

Ans.

(1) \[ 10000 \times (1 + 1)^1 = 20,000 \quad \text{yen} \]
(2) \[ 10000 \times (1 + 1/2)^2 = 22,500 \quad \text{yen} \]
(3) \[ 10000 \times (1 + 1/12)^{12} = 26,130 \quad \text{yen} \]
(4) \[ 10000 \times (1 + 1/365)^{365} = 27,148 \quad \text{yen} \]

[ Note ] Napier’s Constant

\[ \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7182 \]
[Ex.5.1] Find the derivatives of the following functions.

(1) \( y = (\ln x)^2 \) ,  (2) \( y = x \ln x \) ,  (3) \( y = \log_{10} x \)

Pause the video and solve the problem by yourself.
[Ex.5.1] Find the derivatives of (1) \( y = (\ln x)^2 \) and (2) \( y = \ln(x^3 + 1) \)

**Ans.**

(1) \[
\frac{d}{dx} \left( \ln x \right)^2 = 2 \ln x \cdot \frac{d}{dx} \ln x = \frac{2 \ln x}{x}
\]

(2) \[
\frac{d}{dx} (x \ln x) = (x)' \cdot \ln x + x \cdot (\ln x)' = \ln x + x \cdot \frac{1}{x} = \ln x + 1
\]

(3) \[
\log_{10} x = \frac{\ln x}{\ln 10}
\]

\[
\frac{d}{dx} \log_{10} x = \frac{d}{dx} \frac{\ln x}{\ln 10} = \frac{1}{\ln 10} \frac{d}{dx} \ln x = \frac{1}{(\ln 10)x}
\]
Lesson 5
Derivatives of Logarithmic Functions and Exponential Functions

5B
• Derivative of exponential functions
Let $f$ and $g$ be inverse functions. Then

$$y = f(x) \quad \leftrightarrow \quad x = g(y)$$

Differentiate both sides of (1) by $y$ and from the chain rule, we have

$$1 = \frac{df(x)}{dy} = \frac{df(x)}{dx} \frac{dx}{dy} = \frac{df(x)}{dx} \frac{dg(y)}{dy} \quad \text{or} \quad 1 = \frac{df(x)}{dx} \frac{dg(y)}{dy}$$

Therefore

$$\frac{df(x)}{dx} = \frac{1}{\left(\frac{dg(y)}{dy}\right)}$$
Derivative of the Exponential Function

Exponential function of base $e$

$$y = f(x) = e^x \quad \leftrightarrow \quad x = g(y) = \ln y$$

Therefore, from the previous slide we have

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{1}{\frac{dg(y)}{dy}} = \frac{1}{\frac{1}{y}} = y$$

$$\frac{d}{dx}(e^x) = e^x$$

Case $y = a^x$

If $a = e^x$, then $x = \ln a$. Therefore

$$y = a^x = (e^{\ln a})^x = e^{x\ln a}$$

From the Chain Rule

$$\frac{d}{dt}(a^x) = \frac{d}{dt}(e^{x\ln a}) = e^{x\ln a} \frac{d}{dt}(x \ln a) = a^x \ln a$$
[Example 5.3] Find the derivative of the following functions.

(1) \( y = e^{2x} \), \quad (2) \quad y = a^{-2x}

Ans.

(1) Chain rule

\[ y' = e^{2x} \cdot (2x)' = 2e^{2x} \]

(2) Chain rule

\[ y' = \left( a^{-2x} \log a \right) \cdot (-2x)' = -2a^{-2x} \log a \]
[Ex.5.2] Find the derivatives of the following functions.

(1) \( y = x a^x \)  
(2) \( y = 2^{\ln x} \)  
(3) \( y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)

Pause the video and solve the problem by yourself.
[Ex.5.2] Find the derivatives of the following functions.

(1) \( y = xa^x \) 
(2) \( y = 2^{\ln x} \) 
(3) \( y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)

(1) Product rule

\[
y' = (x)'a^x + x(a^x)' = a^x + x \cdot a^x \log a = a^x (1 + x \log a)
\]

(2) Chain rule

\[
y' = 2^{\ln x} \ln 2 \cdot \left( \frac{1}{x} \right) = \frac{2^{\ln x} \ln 2}{x}
\]

(3) Quotient rule

\[
y' = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}
\]