Lesson 3
Differentiation Formulas

3A
- Power Rule
- Linearity Rule
- Product Rule
- Quotient Rule
The Power Rule

\[ \frac{d}{dx} x^n = nx^{n-1} \]

• This rule holds for any real number.

[Examples 3-1] Differentiate the following functions.

1. \( f(x) = \frac{1}{x^2} \)
2. \( f(x) = \sqrt{x^3} \)

1. \[ f'(x) = \frac{d}{dt} \left( x^{-2} \right) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3} \]

2. \[ f'(x) = \frac{d}{dt} \left( \sqrt{x^3} \right) = \frac{d}{dt} \left( x^{\frac{3}{2}} \right) = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2} \sqrt{x} \]
Assume that \( f(x) \) and \( g(x) \) are differentiable functions.

- **Constant Multiple rule**: \((cf)' = cf'\)
- **Sum Rule**: \((f + g)' = f' + g'\)
- **Difference Rule**: \((f - g)' = f' - g'\)

**[Proof of Sum Rule]**

By definition

\[
\left\{ f(x) + g(x) \right\}' = \lim_{h \to 0} \frac{(f(x + h) + g(x + h)) - (f(x) + g(x))}{h}
\]

After rearrangement, we have

\[
\left\{ f(x) + g(x) \right\}' = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f'(x) + g'(x)
\]
### Product Rule

Assume that $f(x)$ and $g(x)$ are differentiable functions.

$$(fg)' = f'g + fg'$$

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**[Examples 3-3]** Find the derivative function of $h(x) = 3x^2(5x + 1)$

**Ans.**

$$h'(x) = 6x(5x + 1) + 3x^2(5) = 45x^2 + 6x$$
The Quotient Rule

\[
\left\{ \frac{f}{g} \right\}' = \frac{f'g - fg'}{g^2}
\]

In particular

\[
\left\{ \frac{1}{g} \right\}' = -\frac{g'}{g^2}
\]

The second one is proved as

\[
\left\{ \frac{1}{g(x)} \right\}' = \lim_{h \to 0} \frac{1}{h} \cdot \left\{ \frac{1}{g(x + h)} - \frac{1}{g(x)} \right\} \\
= \lim_{h \to 0} \left\{ -\frac{g(x+h) - g(x)}{h} \cdot \frac{1}{g(x+h)g(x)} \right\} = -\frac{g'(x)}{\{g(x)\}^2}
\]

(i)

(ii)

Using this, we have

\[
\left\{ \frac{f(x)}{g(x)} \right\}' = \left\{ f(x) \cdot \frac{1}{g(x)} \right\}' = \frac{f'(x)}{g(x)} + f(x) \cdot \frac{-g'(x)}{\{g(x)\}^2} = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}
\]

[Examples 3-4] Compute the derivative function of \( f(x) = \frac{x}{x + 1} \)

Ans.

\[
f'(x) = \frac{1(x + 1) - x(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}
\]
[Exercise 3-1] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) \[ g(x) = \frac{x^3 + 2x^2 + 3x^{-1}}{x} \]  

(2) \[ h(t) = \frac{t^2 - 1}{t - 1} \]

Pause the video and solve the problem by yourself.
[Ex. 3-1] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) \[ g(x) = \frac{x^3 + 2x^2 + 3x^{-1}}{x} \]

Quotient Rule
\[
g'(x) = \frac{(3x^2 + 4x - 3x^{-2})x - (x^3 + 2x^2 + 3x^{-1}) \cdot 1}{x^2} = \frac{2x^3 + 2x^2 - 6x^{-1}}{x^2} = 2x + 2 - 6x^{-3}
\]

Power Rule
\[
g'(x) = (x^2 + 2x + 3x^{-2})' = 2x + 2 - 6x^{-3}
\]

(2) \[ h(t) = \frac{t^2 - 1}{t - 1} \]

Quotient Rule
\[
h'(t) = \frac{2t(t - 1) - (t^2 - 1) \cdot 1}{(t - 1)^2} = \frac{t^2 - 2t + 1}{(t - 1)^2} = \frac{(t - 1)^2}{(t - 1)^2} = 1
\]

Power Rule
\[
h'(t) = (t + 1)' = 1
\]
Lesson 3
Differentiation Formulas

3B
• Chain Rule
• Derivative of Implicit Functions
The Chain Rule

Composite function \( y = f(g(x)) \)

\( y = f(u) \) and \( u = g(x) \)

If \( f(x) \) and \( g(x) \) are differentiable, the next relationship holds.

\[
\frac{df(g(x))}{dx} = \frac{df(g)}{dg} \frac{dg(x)}{dx}
\]

Setting \( u = g(x) \), we may also write this as

\[
\frac{dy}{dx} = f'(u) \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]
[Examples 3-5] Calculate the derivative of \( y = \sqrt{x^3 + 1} \)

Ans.

This is a composite function in the form

\[ f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = x^3 + 1 \]

Since \( f'(u) = \frac{1}{2} u^{-1/2} \) and \( g'(x) = 3x^2 \), we have

\[
\frac{d}{dx} \sqrt{x^3 + 1} = \frac{1}{2} u^{-1/2} (3x^2) = \frac{1}{2} (x^3 + 1)^{-1/2} (3x^2) = \frac{3x^2}{2\sqrt{x^3 + 1}}
\]
Two kinds of function

- Explicit function: \( y = f(x) \)  
  - Example: \( y = x^2 \)
- Implicit function: \( f(x, y) = 0 \)  
  - Example: \( x^2 + y^2 = 1 \)

Two ways to calculate the derivative.

Ex: circle \( x^2 + y^2 = 4 \)

(1) Solve for \( y \) and then differentiate.

\[
y = \pm \sqrt{4 - x^2}
\]

\[
\frac{dy}{dx} = \left( \pm \frac{x}{\sqrt{4 - x^2}} \right) = -\frac{x}{y}
\]
(2) Take derivative of each term and apply the chain rule.

[Ex.] A circle \( x^2 + y^2 = 1 \)

Take the derivative of both sides

\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1)
\]

\[
\therefore 2x + \frac{d}{dx} (y^2) = 0
\]

Applying the chain rule

\[
2x + \frac{d}{dy} (y^2) \frac{dy}{dx} = 0
\]

\[
\therefore 2x + 2y \frac{dy}{dx} = 0
\]

Then

\[
\frac{dy}{dx} = -\frac{x}{y}
\]
[ Exercise 3-2 ] Calculate the derivatives of the following functions

1. \[ y = (x^2 + 2x - 1)^2 \]
2. \[ y = \frac{1}{(3x - 2)^2} \]

Pause the video and solve the problem by yourself.
[ Ex.3-3 ] Calculate the derivatives of the following functions

(1) \( y = (x^2 + 2x - 1)^2 \)

(2) \( y = \frac{1}{(3x - 2)^2} \)

(1) \[ y' = 2(x^2 + 2x - 1)(2x + 2) = 4(x^2 + 2x - 1)(x + 1) \]

(2) \[ y' = \left\{ (3x - 2)^{-2} \right\}' = -2(3x - 2) \cdot 3 = -6(3x - 2) \]
[ Ex.3-4 ] Find the derivative \( \frac{dy}{dx} \) of the function

\[
\frac{x^2}{9} + \frac{y^2}{4} = 1
\]
[ Ex.3-4 ] Find the derivative \( \frac{dy}{dx} \) of the function \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \)

Taking the derivative of both sides with respect to \( x \), we have

\[
\frac{2x}{9} + \frac{2y}{4} \frac{dy}{dx} = 0
\]

Therefore, when \( y \neq 0 \), we have

\[
\frac{dy}{dx} = -\frac{4x}{9y}
\]