Lesson 2
Derivative and Graphs

2A
• Tangent Lines to a Graph of a Function


The straight line

The straight line through \( P(a, b) \) with slope \( m \)

\[
m = \frac{y - b}{x - a}
\]

Therefore

\[
y - b = m(x - a)
\]

The tangent line

The tangent line to the graph of \( y = f(x) \) at point \( (a, f(a)) \) is

\[
y - f(a) = f'(a)(x - a)
\]
Examples

[Examples 2-1] Find an equation of the tangent line to the graph 
\[ y = f(x) = x^2 + 1 \] at \( x = 1 \).

Ans. 

The \( y \)-coordinate at \( x = 1 \) is \( f(1) = 2 \).

Since \( f'(x) = 2x \) we have \( f'(1) = 2 \).

Therefore, 
\[ y - 2 = 2(x - 1) \]
\[ \therefore y = 2x \]
[Examples 2-2] Find the equations and contact points of the tangent lines which contact with $y = x^2 + 4$ and pass $(1, 1)$.

Ans.

Let the contact point be $(a, a^2 + 4)$.

The derivative function $f'(x) = 2x$.

The tangent line is

$$y - (a^2 + 4) = 2a(x - a)$$

Since this line passes $(1, 1)$

$$1 - (a^2 + 4) = 2a(1 - a)$$

$$\therefore a^2 - 2a - 3 = 0, \quad \therefore a = -1, 3$$

When $a = -1$

contact point $(-1, 5)$, tangent line $y = -2x + 3$

When $a = 3$

contact point $(3, 13)$, tangent line $y = -6x + 5$
Behavior of graphs based on its derivatives

In open domains A and C:

\[ f'(x) < 0 \rightarrow f(x) \text{ is increasing.} \]

In open domain B:

\[ f'(x) > 0 \rightarrow f(x) \text{ is decreasing.} \]
**Example**

**Examples 2-3** Investigate the change of the function \( y = x^3 + 3x^2 - 2 \) and illustrate the graph.

**Derivative function** \( y' = 3x^2 + 6x \)

**Horizontal tangent line**

\[
y' = 3x(x + 2) = 0 \quad x = -2, \ 0
\]

\( y' > 0 \) in \( x < -2, \ x > 0 \)

\( y' < 0 \) in \( -2 < x < 0 \)

| \( x \) | ..... | -2 | ..... | 0 | ..... |
|---|---|---|---|---|
| \( y' \) | + | 0 | - | 0 | + |
| \( y \) | \( \nearrow \) | 2 | \( \searrow \) | -2 | \( \nearrow \) |

y-intercept : \((0, 2)\)

x-intercepts: \( y = x^3 + 3x^2 - 2 = (x + 1)(x^2 + 2x - 2) = 0 \)

\((-1, 0), \ (-1+\sqrt{3}, 0) \ (-1-\sqrt{3}, 0)\)
[Exercise 2-1] About the graph \( y = x^3 - 2x^2 - 3x \), answer the following questions.

1. Find the equation of the tangent line at (-1, 0).
2. Find the cross-point of the graph and this tangent line.

(Hint: The cross-point of \( y = f(x) \) and \( y = ax + b \) is given by the roots of \( f(x) = ax + b \)).

Ans.

Pause the video and solve the problem by yourself.
[Exercise 2-1] About the graph $y = x^3 - 2x^2 - 3x$, answer the following questions.

1. Find the equation of the tangent line at (-1, 0).
2. Find the cross-point of the graph and this tangent line.

**Ans.**

The derivative function $y' = 3x^2 - 4x - 3$

$\therefore f''(-1) = 4$

(1) Tangent line

$y - 0 = 4(x + 1) \quad \therefore y = 4x + 4$

(2) Cross point of the tangent line and the curve

$x^3 - 2x^2 - 3x = 4x + 4$

By factoring

$(x + 1)^2(x - 4) = 0$

Therefore,

$x = 4$

The cross-point is (4, 20).
Lesson 2
Derivative and Graphs

2B

• Local Extrema
• Global Extrema
• Second derivative test
Local Extrema

• The function has a **local maximum** (**local minimum**) at $x = c$
  
  if $f(c)$ is the **maximum** (**minimum**) value in a neighborhood around $c$.

• If $f(c)$ is a local extrema then $f'(c) = 0$
Global Extrema

- Absolute maximum (or global maximum) is the maximum value in the whole domain \([a, b]\).
- Absolute minimum (or global minimum) is the minimum value in the whole domain \([a, b]\).
(1) That the slope is zero \( f'(c) = 0 \) does not necessarily mean that the point is a local max./min. point.

(2) A number \( c \) in the domain of \( f(x) \) is called a critical point if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.
[Examples 2-4] Find the maximum and minimum values of \( f(x) = 2x^3 + 3x^2 - 12x + 1 \) in the domain \(-1 \leq x \leq 2\).

**Ans.**

1. Find extrema:
   \[ f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) \]
   \[ x = -2, \ 1 \]
   \[ f(1) = -6 \]

2. Find the values at the boundaries:
   \[ f(-1) = 14, \ f(2) = 5 \]

3. Find \( y \)-intercept:
   \[ y = f(0) = 1 \]

4. Make a table:
<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>( \cdots )</th>
<th>1</th>
<th>( \cdots )</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>14</td>
<td>( \downarrow )</td>
<td>Local Min.</td>
<td>-6</td>
<td>( \uparrow )</td>
</tr>
</tbody>
</table>

5. Illustrate the graph

6. Find the max. and min. values:
   Max. value 14 (at \( x = -1 \)), Min. value -6 (at \( x = 1 \))
Shape of a graph and Second Derivative

Shape of a graph

Local maximum

Concave down

\[ f'(x) > 0 \rightarrow f'(a) = 0 \rightarrow f''(x) < 0 \]

At \( x = a \), slope decreases

\[ f''(a) < 0 \]

Local minimum

Concave up

\[ f'(x) < 0 \rightarrow f'(b) = 0 \rightarrow f''(x) > 0 \]

At \( x = a \), slope increases

\[ f''(b) > 0 \]
Second Derivative Test

Second derivative test

Suppose that \( f'(c) = 0 \) at \( x = c \).

- If \( f''(c) > 0 \) then \( f(x) \) is a local minimum at \( x = c \).
- If \( f''(c) < 0 \) then \( f(x) \) is a local maximum at \( x = c \).
- If \( f''(x) = 0 \) then the situation is inconclusive.

Inflection Point

The point where the concavity of the graph changes from up to down or vice versa is called an inflection point.
[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When does the volume of this cup become maximum?

Ans.

Pause the video and solve the problem by yourself.
Answer to the Exercise

[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When does the volume of this cup becomes maximum?

Ans. The side length of the squares cut off from the corner: \( x \)

Then,
- the volume
- Condition \( 0 < x < 6 \)
- The derivative \( V' = 12(x - 2)(x - 6) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \ldots )</th>
<th>2</th>
<th>( \ldots )</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V' )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( V )</td>
<td>128</td>
<td>128</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ V = (12 - 2x)^2 x = 4(x^3 - 12x^2 + 36x) \]