Course II : Calculus
What is Calculus?

*Calculus* is a branch of mathematics.
- functions, • limits, • derivatives, • integrals, • power series

*Calculus* is the study of change.
  cf. • *Geometry* is the study of shape.
  • *Algebra* is the study of operation.

*Calculus* is a gateway to advanced mathematics.
- We must study and understand completely.

*Calculus* has wide applications in
- science, • engineering, • economics, • biology

*Calculus* has two branches
- differential calculus, • integral calculus,
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Lesson 1
Limit of Functions and Derivatives

1A
• Limit of a function
Definition of a Limit

If a function $f(x)$ can be made to be as close to $L$ as desired by making $x$ sufficiently close to $a$, we say that “the limit of $f(x)$, as $x$ approaches $a$, is $L$” and we write as follows

$$\lim_{x \to a} f(x) = L$$

We can also write $f(x) \to L$ as $x \to a$ and read “$f(x)$ approaches $L$ as $x$ approaches $a$.”
[Example] \( f(x) = x^2 \)

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<th>( f(x) )</th>
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The limit of \( f(x) = x^2 \) as \( x \) approaches 2 is 4
Several Comments about the Limit

○ For the limit of a function to exist, the left-hand and right-hand limits must be equal, that is

\[ \lim_{{x \to a^-}} f(x) = L \quad \text{and} \quad \lim_{{x \to a^+}} f(x) = L \]

Limit does not exit at \( x = 2 \)

○ \( \lim_{{x \to a}} f(x) = L \) is not always equal to \( f(a) \)
Example 1.1 Find the limit value of the following function.

\[
\lim_{x \to 1} \frac{x^2 - x}{x - 1}
\]

Ans.

\[
\lim_{x \to 1} \frac{x^2 - x}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{x - 1} = \lim_{x \to 1} x = 1
\]

○ Even if the function has not a value at \( x = a \), the limit may exist.

Indeterminate Form

The forms \( 0^0, \frac{0}{0}, 1^\infty, \infty - \infty, \frac{\infty}{\infty}, 0 \times \infty, \infty^0 \), etc. are called indeterminate forms because they do not give enough information to determine values.
Example

Example 1.2  Find the limit value of the following function.

\[ \lim_{x \to 1} \frac{\sqrt{x - 1} - 1}{x} \]

Ans.

\[ \lim_{x \to 1} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x \to 1} \frac{\left(\sqrt{x + 1} - 1\right)\left(\sqrt{x + 1} + 1\right)}{x\left(\sqrt{x + 1} + 1\right)} = \lim_{x \to 1} \frac{(x + 1) - 1}{x\left(\sqrt{x + 1} + 1\right)} \]

\[ = \lim_{x \to 1} \frac{1}{\left(\sqrt{x + 1} + 1\right)} = \frac{1}{2} \]

Means to find a limit of an Indeterminate Form 0/0

(1) Case of Polynomial \rightarrow \text{Factor them}

(2) Case of Irrational Function \rightarrow \text{Multiply the conjugate}
Example 1.3 Determine the values of $a$ and $b$ so that the following expression holds.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + ax + b} = 1$$

Ans.

When $x \to 1$, then $x^2 + x - 2 \to 0$ and $x^2 + ax + b \to 1 + a + b$

In order for limit to exist, $1 + a + b$ must be zero. \[ \therefore b = -a - 1 \]

Substituting this, we have

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + ax + b} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(x+a+1)} = \lim_{x \to 1} \frac{x+2}{x+a+1} = \frac{3}{a+2}$$

Therefore,

$$\frac{3}{a+2} = 1 \quad \therefore \quad a = 1 \text{ and } b = -2$$
[Exercise 1.1] Determine the values of \( a \) and \( b \) so that the following relationship holds.

\[
\lim_{{x \to 1}} \frac{ax^2 + bx + 1}{x - 1} = 3
\]

Ans.

Pause the video and try to solve by yourself.
[ Exercise 1.1 ] Determine the values of $a$ and $b$ so that the following relationship holds.

\[
\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = 3
\]

Ans.

When $x \to 1$, then $x - 1 \to 0$ and $ax^2 + bx + 1 \to a + b + 1$

In order to exist a limit value 1, $a + b + 1 = 0$ \quad \therefore \quad b = -a - 1

Substituting this, we have

\[
\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(ax - 1)}{x - 1} = \lim_{x \to 1} (ax - 1) = a - 1
\]

Therefore,

\[a - 1 = 3\]  \quad \text{Namely,} \quad a = 4 \quad \text{and} \quad b = -5
Lesson 1
Limit of Functions and Derivatives

1B
• Derivatives of Functions
Average Rate of Change

The slope

\[ \Delta x = x_2 - x_1 \]

\[ \Delta y = y_2 - y_1 \]

The slope

\[ \frac{\Delta y}{\Delta x} \]

Increments

Average Rate of Change

\[ \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{f(b) - f(a)}{b - a} \]
Definition of a Derivative

**Derivative**

The slope at point A (the tangent line T) can be obtained by making point B approach point A.

\[
\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a)
\]

This is called the derivative of \( f(x) \) at \( a \).

Putting \( b = a + h \), we also have

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)
\]

That makes sense!
How to Find the Derivative

[Example 1-4 ] About the function \( f(x) = x^2 \)

(1) Find the average rate of change between \( x = 1 \) and \( x = 2 \).

(2) Find the instantaneous rate of change at \( x = a \).

(3) Find the point where the instantaneous rate of change is equal to the average rate of change between \( x = 1 \) and \( x = 2 \).

\[
\begin{align*}
\text{Ans.} \quad (1) \quad & \frac{f(2) - f(1)}{2 - 1} = 4 - 1 = 3 \\
(2) \quad f'(a) &= \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = \lim_{h \to 0} (2a + h) \\
&= 2a \\
(3) \quad \text{Using the results of (1) and (2), we put } \quad 2a = 3 \\
&\therefore \quad a = \frac{3}{2}
\end{align*}
\]
Derivative as a Function

Let the number \( a \) varies and replace it by \( x \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\( f'(x) \) is called the derivative of \( f(x) \) or the derivative function of \( f(x) \) (because it has been “derived” from \( f(x) \)).

Alternative notation

\[
f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)
\]

[NOTE]
The definition \( \frac{dy}{dx} \) is read as: “the derivative with respect to \( x \),” “\( \frac{dy}{dx} \),” “\( dy \)” over “\( dx \)” or simply “\( dy \)” “\( dx \)”
[Example 1-4] Find the derivative function of

(1) \( f(x) = x \)  \hspace{1cm} (2) \( f(x) = x^2 \)  \hspace{1cm} (3) \( f(x) = x^3 \).

**Ans.**

**Definition** \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

(1) \( f'(x) = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} (1) = 1 \)

(2) \( f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x \)

(3) \( f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \)

**Formula**

\[
\frac{d}{dx} (x^n) = nx^{n-1}
\]
Higher Derivatives

Since \( f'(x) \) is a function, it also has its own derivative which is denoted by

\[
\frac{d^2 y}{dx^2} = f''(x) = f^{(2)}(x) \quad : \text{The second derivative function}
\]

We can continue

\[
\frac{d^3 y}{dx^3} = f'''(x) = f^{(3)}(x) \quad : \text{The third derivative function}
\]

The process of finding a derivative function is called differentiation.
[Example 1-5 ] If \( f(x) = x^3 - x \), find and interpret \( f''(x) \)

Using the formula \( \frac{d}{dx}(x^n) = nx^{n-1} \), we get

\[
f'(x) = 3x^2 - 1 \]
\[
f''(x) = 6x \]

These derivatives are illustrated in the Right-hand side.

- \( f''(x) \) is the slope of the curve \( y = f'(x) \)
- \( f''(x) \) is the rate of change of \( y = f'(x) \)
Exercise

[ Exercise 1.2 ] Function \( f(x) = x^3 + ax^2 + bx + c \)
Satisfy the conditions \( f(1) = 3 \), \( f(0) = 1 \) and
\( f'(-1) = 16 \). Find the constants \( a \), \( b \) and \( c \).

Ans.

Pause the video and try to solve by yourself
[Exercise 1.2] Function \( f(x) = x^3 + ax^2 + bx + c \)
Satisfy the conditions \( f(1) = 3 \), \( f(0) = 1 \) and \( f'(-1) = 16 \). Find the constants \( a \), \( b \) and \( c \).

Ans.

The derivative function is
\[
 f'(x) = 3x^2 + 2ax + b 
\]

Given condition
\[
 f(1) = 1 + a + b + c = 3 
\]
\[
 f(0) = c = 1 
\]
\[
 f'(-1) = 3 - 2a + b = 16 
\]

From these equations
\[
 a = -4, \ b = 5, \ c = 1 
\]