Lesson 15
Graphs and Equations (II)

15A
- Circle
- Circle and a Straight Line
Circle

Condition
\[ \sqrt{(x - a)^2 + (y - b)^2} = r \quad (\text{where } r > 0) \]

Standard expression
\[ (x - a)^2 + (y - b)^2 = r^2 \]

By expanding this, we obtain
\[ x^2 - 2ax + y^2 - 2by + a^2 + b^2 = r^2 \]

General expression
\[ x^2 + y^2 + lx + my + n = 0 \quad (\text{where } l^2 + m^2 > 4n) \]
Terminologies

Lines
- Chord
- Arc
- Diameter
- Radius
- Tangent

Slices
- Sector
- Segment

Special types of sector
- Quadrant
- Semicircle
[Example 15.1] Answer the following questions.

(1) Find the circle which passes points $A(-1, 7), B(2, -2)$ and $C(6, 0)$.

(2) Find the center of this circle.

Ans.

$$x^2 + y^2 + lx + my + n = 0$$

(1) Because the circle passes points $A, B, C$

$$-l + 7m + n = -50$$
$$2l - 2m + n = -8$$
$$6l + n = -36$$

Therefore

$$l = -4, \ m = -6, \ n = -12$$ and $$x^2 + y^2 - 4x - 6y - 12 = 0$$

(2) This equation is rearranges to the standard form as follows.

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) - 4 - 9 - 12 = 0$$

$$(x - 2)^2 + (x - 3)^2 = 5^2$$

Therefore, the center is $(2, 3)$
Relationship between a Circle and a Straight Line

Three cases

\[ d < r \]

- (a) Cross

\[ d = r \]

- (b) Contact

\[ d > r \]

- (c) No contact

Simultaneous Equation Problem

Circle \[ x^2 + y^2 + lx + my + n = 0 \]

Line \[ y = px + q \]

\[ \begin{align*}
\text{Discriminant} & \quad D = b^2 - 4ac \\
(a) \ \text{Cross} & \quad D > 0 \\
(b) \ \text{Contact} & \quad D = 0 \\
(c) \ \text{No contact} & \quad D < 0
\end{align*} \]
[Example 15.2] Find the coordinates of the common points of a circle and a straight line in the following.

(1) Circle \( x^2 + y^2 = 5 \) (i) and line \( y = x - 1 \) (ii).

(2) Circle \( x^2 + y^2 = 5 \) (iii) and line \( 2x - y + 5 = 0 \) (iv).

**Ans.**

(1) Substituting Eq.(ii) into Eq.(i), we have

\[
x^2 - x - 2 = (x + 1)(x - 2) = 0
\]

\[
\therefore x = -1, \quad 2 \quad \therefore y = -2, \quad 1
\]

The common points are \((-1, -2)\) and \((2, 1)\).

(2) Substituting Eq.(iv) into Eq.(iii), we have

\[
x^2 + 4x + 4 = (x + 2)^2 = 0
\]

\[
\therefore x = -2, \quad \therefore y = 1
\]

Therefore, the contact point is \((-2, 1)\).
[Exercise 15.1] The straight line $y = -2x + 1$ crosses with the circle $x^2 + y^2 = 1$. Find the chord length.

Ans.

Pause the video and solve the problem by yourself.
Answer to the Exercise

[Exercise 15.1] The straight line \( y = -2x + 1 \) crosses with the circle \( x^2 + y^2 = 1 \). Find the chord length.

**Ans.**

By eliminating \( y \), we have

\[
x^2 + (-2x + 1)^2 = 1
\]

\[
5x^2 - 4x = 5x(x - \frac{4}{5}) = 0
\]

\[
\therefore x = 0, \frac{4}{5}
\]

Then, the two cross points are

\((0, 1), (\frac{4}{5}, \frac{-3}{5})\)

The chord length is

\[
\sqrt{\left(0 - \frac{4}{5}\right)^2 + \left(1 + \frac{3}{5}\right)^2} = \frac{4\sqrt{5}}{5}
\]
Lesson 15
Graphs and Equations (II)

15B
• Tangent line to a Circle
• Line through a cross point
• Circle through cross points
• Region defined by inequalities
Tangent Line to a Circle

Tangent Line

• A tangent line to a circle intersects at a single point T.
• The radius of a circle is perpendicular to the tangent line.

• The equation of the tangent line is given by

\[ y - y_1 = -\frac{x_1}{y_1} (x - x_1) \]

That is,

\[ x_1 x + y_1 y = r^2 \]

Slope

\[ m = \frac{y_1}{x_1} \]

Uhh…. What?
Example

[Example 15.3] Find the tangent lines to circle \( x^2 + y^2 = 2 \), which passes through point \((3, 1)\).

**Ans.** Put tangent line \( x_1x + y_1y = 2 \)

Since his line passes \((3, 1)\)

\[ 3x_1 + y_1 = 2 \] (i)

Since \((x_1, y_1)\) is on the given circle

\[ x_1^2 + y_1^2 = 2 \] (ii)

From (i) and (ii), we have

\[ 5x_1^2 - 6x_1 + 1 = (5x_1 - 1)(x_1 - 1) = 0 \]

\[ x_1 = \frac{1}{5}, 1 \quad \text{Therefore} \quad (x_1, y_1) = \left( \frac{1}{5}, \frac{7}{5} \right), (1, -1) \]

The tangent lines are

\[ \frac{1}{5}x + \frac{7}{5}y = 2 \quad x - y = 2 \]
Line and Circle Passing Through Cross Points

**Line**

Two lines:  
\[ a_1 x + b_1 y + c_1 = 0 \]  \( \text{①} \)  
and  
\[ a_2 x + b_2 y + c_2 = 0 \]  \( \text{②} \)

The line which passes the cross point is  
\[ k \cdot (a_1 x + b_1 y + c_1) + a_2 x + b_2 y + c_2 = 0 \]

**Circle**

Two circles:

\[ f(x, y) = x^2 + y^2 + l_1 x + m_1 y + n_1 = 0 \]  \( \text{①} \)  
\[ g(x, y) = x^2 + y^2 + l_2 x + m_2 y + n_2 = 0 \]  \( \text{②} \)

The circle which passes the cross points  
\[ k \cdot f(x, y) + g(x, y) = 0 \]
[Example 15.4] Find the expression of a circle which passes the cross points of two circles \( x^2 + y^2 - 2x - 4y + 4 = 0 \) and \( x^2 + y^2 - 4x - 2y = 0 \), and also passes point (1, 2).

**Ans.**

The line which passes the cross points.

\[
k(x^2 + y^2 - 2x - 4y + 4) + x^2 + y^2 - 4x - 2y = 0
\]

Substituting the coordinate of the point (1, 2), we have

\[
- k - 3 = 0 \quad \therefore k = -3
\]

Then

\[
- 3(x^2 + y^2 - 2x - 4y + 4) + x^2 + y^2 - 4x - 2y = 0
\]

The answer is

\[
(x^2 + y^2) - x - 5y + 6 = 0
\]
Region Defined by Inequalities

**Line**

Straight line $l: \ y = mx + n$

**Circle**

Circle : $(x - a)^2 + (y - b)^2 = r^2$

$(x - a)^2 + (y - b)^2 > r^2$

$(x - a)^2 + (y - b)^2 < r^2$
[Example 15.4] Illustrate the region given by the following expression.

\[ x^2 - 4x + y^2 - 6y + 9 < 0 \]

**Ans.** First, we rearrange the expression to the following standard form.

\[ (x - 2)^2 + (y - 3)^2 < 2^2 \]

Therefore, the boundary is given by a circle with radius 2 and with center (2, 3).

The answer is given by the shaded region.
Exercise

[Exercise 15.2] Illustrate the region which satisfies the following inequality.

\[(y - x + 3)(x^2 + y^2 - 12x - 6y + 20) < 0\]

Ans.

Pause the video and solve the problem by yourself.
Exercise

[Exercise 15.2] Illustrate the region which satisfies the following inequality.

\[(y - x + 3)(x^2 + y^2 - 12x - 6y + 20) < 0\]

Ans. \(AB < 0 \iff A > 0, \ B < 0 \text{ or } A < 0, \ B > 0\)

We add the following two regions. After rearrangement, we have

(1) \(y > x - 3\) and \((x - 6)^2 + (y - 3)^2 < 5^2\)
(2) \(y < x - 3\) and \((x - 6)^2 + (y - 3)^2 > 5^2\)

Let the boundary \(y = x - 3\) be \(l\)
and the boundary
\[(x - 6)^2 + (y - 3)^2 = 5^2\] be \(C\).

Then,
the range (1) is above \(l\) and inside of \(C\),
the range (2) is below \(l\) and outside of \(C\).