Lesson 10
Inverse Functions

10A
• Inverse functions
• Exponential Function and Logarithmic Function
Example: Consider \( y = f(x) = \frac{1}{2}x + 1 \)

The inverse function is given by exchanging \( x \) and \( y \).

\[
x = \frac{1}{2}y + 1 \quad \text{that is} \quad y = g(x) = 2x - 2
\]

The inverse function \( g(x) \) brings \( f(1) \) back to 1.

\[
1 = g(1.5) \quad 1.5 = f(1)
\]
Inverse Function

Definition of Inverse Functions

Functions $f(x)$ and $g(x)$ are inverses of one another if: 

$f(g(x)) = x$ and $g(f(x)) = x$

for all values of $x$ in their respective domains.

How to Find Inverse Function

1. Replace the variables.
   \[ y = f(x) \rightarrow x = f(y) \]

2. Rearrange the expression.
   \[ x = f(y) \rightarrow y = g(x) \]

3. Determine that the domain of $g(x)$ is the same as the range of $f(x)$.
Inverse Function – Example(2)

Parabola

Not inverse

Because (b) is not a function

Inverse function of each other
Exponential Function & Logarithmic Function

Exponential function

\[ y = a^x \quad (a > 0, \ a \neq 0) \]

- \( a > 1 \)
- \( 0 < a < 1 \)

The domain is the positive real number.
- The range is the whole real number.
- The graph always passes the point \((1, 0)\)

The inverse function of \( y = a^x \) is called \textbf{a logarithmic function} and written as \( y = \log_a x \)

- The domain is the positive real number.
- The range is the whole real number.
- The graph always passes the point \((1, 0)\)
Example 1. Illustrate the function \( y = -\sqrt{1-x} \) and its inverse function on the given coordinate plane.

\[
\begin{array}{c|ccccc}
\hline
x & 1 & 0 & -1 & -2 & -3 \\
\hline
y & 0 & 1 & \sqrt{2} & \sqrt{3} & 2 \\
\hline
\end{array}
\]

- Take symmetrical points about \( x=y \), that is,
- replace their values

\[
\begin{array}{c|cccccc}
\hline
x & 0 & 1 & 2 & 3 \\
\hline
y & 1 & 0 & -1 & -2 \\
\hline
\end{array}
\]
Exercise 1. (1) Illustrate the function

\[ y = \frac{x}{x + 2} \quad (x \geq 0) \]

(2) Illustrate the inverse function by mapping symmetrically about \( y = x \)

(3) Find the domain of the inverse function.

(4) Find the expression of the inverse function by replacing \( x \) and \( y \).

Ans.

Pause the video and solve the problem.

\[
\begin{array}{c|cccccccc}
\hline
& y & 4 & 3 & 2 & 1 & -1 & -2 & -3 & -4 \\
\hline
x & 0 & 1 & 2 & 3 & 4 & & & & \\
\hline
\end{array}
\]
Answer to the Exercise

Exercise 1. (1) Illustrate the function

\[ y = \frac{x}{x + 2} \quad (x \geq 0) \]

(2) Illustrate the inverse function by mapping symmetrically about \( y = x \)

(3) Find the domain and the range of the inverse function.

(4) Find the expression of the inverse function by replacing \( x \) and \( y \).

Ans. (1) This expression becomes

\[ y = 1 - \frac{2}{x + 2} \]

(2) See the figure.

(3) From the figure

\begin{align*}
\text{Domain:} & \quad 0 \leq x < 1 \\
\text{Range:} & \quad y \geq 0
\end{align*}

(4) \( x = \frac{y}{y + 2} \)

\[ \therefore \quad y = \frac{-2x}{x - 1} \]
Lesson 10
Hyperbolic Functions

10B
• Catenary
• Hyperbolic Function
• Why Do We Call Them “Hyperbolic”?
Catenary

Hanging chain (Catenary)

Hyperbolic Function

\[ \sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]
Trigonometric functions

\[\tan x = \frac{\sin x}{\cos x}\]
\[\cos^2 x + \sin^2 x = 1\]
\[1 + \tan^2 x = \frac{1}{\cos^2 x}\]

Hyperbolic functions

\[\tanh x = \frac{\sinh x}{\cosh x}\]
\[\cosh^2 x - \sinh^2 x = 1\]
\[1 - \tanh^2 x = \frac{1}{\cosh^2 x}\]
Why Do We Call Them “Hyperbolic”?

Comparison of Two Parametric Expressions

\[(x,y) = (\cos t, \sin t)\]

Eliminating the parameter, we have

\[x^2 + y^2 = 1\]

This expression represents a circle.

\[(x,y) = (\cosh t, \sinh t)\]

From the previous slide, we have

\[x^2 - y^2 = 1\]

This expression represents a hyperbola.

- Based on the similarity in shape, we call them hyperbolic functions.
- Based on the similarity in character, we use the symbols \(\sinh x\) etc.
Example 2. Prove the following addition formula.

\[
\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y
\]

**Ans.**

\[
\cosh x \cosh y + \sinh x \sinh y = \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) + \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right)
\]

\[
= \frac{1}{4} \left( e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y} \right) + \frac{1}{4} \left( e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y} \right)
\]

\[
= \frac{1}{2} \left( e^{x+y} + e^{-x-y} \right) = \cosh(x + y)
\]
Exercise

Exercise 2. Prove the following formulae.

(1) \( \cosh^2 x - \sinh^2 x = 1 \)

(2) \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)

Ans.

Pause the video and solve the problem.
Exercise 2. Prove the following formulae.

(1) \( \cosh^2 x - \sinh^2 x = 1 \)

(2) \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)

Ans.

(1) \( \cosh^2 x - \sinh^2 x = \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2 \)

\[ = \frac{1}{4} (e^{2x} + 2 + e^{-2x}) - \frac{1}{4} (e^{2x} - 2 + e^{-2x}) = 1 \]

(2) \( \sinh x \cosh y + \cosh x \sinh y = \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^y + e^{-y}}{2} \right) + \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^y - e^{-y}}{2} \right) \)

\[ = \frac{1}{4} (e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y}) + \frac{1}{4} (e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}) \]

\[ = \frac{1}{2} (e^{x+y} - e^{-x-y}) = \sinh(x + y) \]