Lesson 9
Rational Function and Irrational Function

9A
- Linear Rational Function
- Simple Rational Function
- Standard Form
- Inequality
Rational Function

\[ y = \frac{g(x)}{h(x)} \]

where \( g(x) \) and \( h(x) \) are polynomials

Linear Rational Function

\[ y = \frac{ax + b}{cx + d} \quad (ad - bc \neq 0, \ c \neq 0) \]

In the case of \( ad - bc = 0 \)

\[ \therefore \frac{a}{c} = \frac{b}{d} (\equiv k), \ \therefore a = ck, \ b = dk, \ \therefore y = \frac{ckx + dk}{cx + d} = k \quad : \text{Constant} \]

In the case of \( c = 0 \)

\[ y = \frac{a}{d} x + \frac{b}{d} \quad : \text{Linear function} \]
The Simplest Rational Function

\[ y = \frac{k}{x} \]

- Two variables are inversely proportional.

- The horizontal asymptote is \( x = 0 \) / the vertical asymptote is \( y = 0 \)
- The graph is an equilaterial hyperbola.
Function \[ y = \frac{ax + b}{cx + d} \]

Change to the standard form

\[ y = \frac{ax + b}{cx + d} \rightarrow y = \frac{k}{x - p} + q \]

- The vertical asymptote is \( x = p \)
- The horizontal asymptote is \( y = q \).

- The domain is \( x \neq p \); the range is \( y \neq q \).
Example 1. Illustrate the rational function \( y = \frac{3x - 6}{x - 1} \), following each steps.

1. **[Step 1]** Convert the function to the standard form.

2. **[Step 2]** Find the horizontal and vertical asymptotes.

3. **[Step 3]** Find the x-intercept and y-intercept.

4. **[Step 4]** Illustrate the graph.

**Ans.**

1. By rearranging, it becomes
   \[
   y = \frac{3x - 3 - 3}{x - 1} = 3 - \frac{3}{x - 1}
   \]

2. The horizontal asymptote is \( y = 3 \)
   
   The vertical asymptote is \( x = 1 \)

3. By putting \( y = 0 \), the x-intercept is \( x = 2 \)
   
   By putting \( x = 0 \), the y-intercept is \( y = 6 \)

4. See the right figure
Example 2. Solve the following inequality \[ \frac{3x - 6}{x - 1} \geq -x + 6 \]

Ans.

We utilize the result of Example 1.

The cross points of \[ y = \frac{3x - 6}{x - 1} \]
and \[ y = -x + 6 \]
are

\[
\frac{3x - 6}{x - 1} = -x + 6 \quad \therefore 3x - 6 = (x - 1)(-x + 6)
\]

\[
\therefore x(x - 4) = 0 \quad \therefore x = 0, \ x = 4
\]

From the figure and the cross points, we have \[ 0 \leq x < 1, \ 4 \leq x \]
**Exercise 1.** Solve the inequality \( \frac{x + 1}{x - 1} > x + 1 \)

**Ans.**

Pause the video and solve the problem.
Exercise 1. Solve the inequality \( \frac{x+1}{x-1} > x+1 \)

Ans.

- We consider two functions \( y = \frac{x+1}{x-1} = \frac{2}{x-1} + 1 \) and \( y = x + 1 \).
- The horizontal asymptote and vertical asymptote of the former are \( y = 1 \) and \( x = 1 \), respectively.
- The x- and y-intercepts of the former are \( x = -1 \) and \( y = -1 \), respectively.
- The x- and the y-intercepts of the latter are \( x = -1 \) and \( y = 1 \), respectively.
- The cross-points are \( \frac{x+1}{x-1} = x + 1 \) \( \therefore (x+1)(x-2) = 0 \) \( \therefore x = -1, \ 2 \)
- From the figure, the solution is \( x < -1, \ 1 < x < 2 \)
Lesson 9
Exponential Functions

9B
• Irrational Functions
• Standard Form
• Inequality
Irrational Function

• There is no rigorous definition of irrational function.
• We can say that irrational function is the one that cannot be written as the quotient of two polynomials (but this definition is not used.).
• Customary, a function which include variables in the root is called an irrational function.

Example
Function

Fundamental Form

\[ y = \sqrt{ax} \]

- **Domain** \( x \leq 0 \)
- **Range** \( y \geq 0 \)

- **Domain** \( x \geq 0 \)
- **Range** \( y \geq 0 \)
Function

Change to the standard form

\[ y = \sqrt{ax + b + q} \]

\[ \rightarrow \]

\[ y = \sqrt{a(x - p) + q} \]

That wasn’t so hard!
Example 3. Answer the following questions

(1) Illustrate the graphs of the functions $y = \sqrt{2 - x}$ and $y = -x + \sqrt{2}$

(2) Solve the equation $\sqrt{2 - x} = -x + \sqrt{2}$

(3) Solve the inequality $\sqrt{2 - x} \geq -x + \sqrt{2}$

Ans. (1) See the right figure

(2) $\sqrt{2 - x} = -x + \sqrt{2}$ \quad \therefore \quad 2 - x = \left(-x + \sqrt{2}\right)^2$

\quad \therefore \quad x(x - 2\sqrt{2} + 1) = 0 \quad \therefore \quad x = 0, \quad 2\sqrt{2} - 1$

From the figure, the root is only $x = 0$

(3) From (1) and (2), we have $0 \leq x \leq 2$

[Note] We must check that the obtained roots really satisfy the equation.
Exercise 2. Solve the inequality \( \sqrt{x + 2} > x \)

Ans.

Pause the video and solve the problem.
Exercise 2. Solve the inequality $\sqrt{x + 2} > x$

Ans.

- We consider two functions $y = \sqrt{x + 2}$ and $y = x$.

- The x- and the y-intercepts of the former are $x = -2$ and $y = \sqrt{2}$, respectively.

- The cross-points of these functions are
  
  $\sqrt{x + 2} = x \quad \therefore \quad x + 2 = x^2$

  $\therefore \quad (x + 1)(x - 2) = 0 \quad \therefore \quad x = -1, \quad x = 2$

Only $x = 2$ satisfies this condition.

- From the figure, the solution is $-2 \leq x < 2$

(Note the boundaries.)