Lesson 7
Complex Numbers

7A
• Solutions of a Quadratic Equation
• Complex Number
Solutions of a Quadratic Equation

[ Review ]

The quadratic Eq. \( ax^2 + bx + c = 0 \)

Solution

If \( D = b^2 - 4ac > 0 \)

- there are two real distinct roots \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \alpha \\ \beta \end{cases} \)

If \( D = b^2 - 4ac = 0 \)

- there is one real double root are \( x = -\frac{b}{2a} \)

If \( D = b^2 - 4ac < 0 \)

- there is no real root. (The equation is unsolvable.)
Introducing a New Number

We introduce an imaginary number

\[ i = \sqrt{-1} \]

which is a theoretical number equal to the square root of -1.

**Quadratic formula**

The roots of the quadratic equation \( ax^2 + bx + c = 0 \) are

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

If \( D = b^2 - 4ac > 0 \), there are two distinct real roots.
If \( D = b^2 - 4ac = 0 \), there is a double root.
If \( D = b^2 - 4ac < 0 \), there are two distinct imaginary roots
Complex Number

Complex number

\[ a + bi \]

( \(a\) : real part, \(b\) : imaginary part, \(i\) : imaginary unit)

Special Cases

\(a\) : real number (case of \(a \neq 0, \ b = 0\))

\(a + bi\) : imaginary number (case of \(b \neq 0\))

\(bi\) : pure imaginary number (case of \(a = 0, \ b \neq 0\))

Complex number = real number + imaginary number
Operation on Complex Numbers

Equality

\[ a + bi = c + di \quad \text{equals} \quad a = c \quad \text{and} \quad b = d \]
\[ a + bi = 0 \quad \text{equals} \quad a = 0 \quad \text{and} \quad b = 0 \]

Addition

\[ (a + bi) + (c + di) = (a + c) + (b + d)i \]

Subtraction

\[ (a + bi) - (c + di) = (a - c) + (b - d)i \]

Multiplication

\[ (a + bi)(c + di) = (ac - bd) + (ad - bc)i \]

Division

\[ \frac{c + di}{a + bi} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}i \]  

(refer to next slide)

In the calculation, deal with \( i \) just as you would do with \( x \), but replace \( i^2 \) by \(-1\).
Complex Conjugates

A complex number \( a - bi \) is called a complex conjugate of \( a + bi \).

the same real part

equal magnitude and opposite signs.

A notation \( z(=a+bi) \) is sometimes used to express a complex number and \( \bar{z}(=a-bi) \) is used to express its complex conjugate.

[ Note ] One merit of complex conjugates

Multiplication of complex conjugates produces a real number

\[
\overline{z}z = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2
\]

Therefore, it is useful in the calculation of division

<Example>

\[
\frac{(c + di)}{(a + bi)} = \frac{(c + di)(a - bi)}{(a + bi)(a - bi)} = \frac{(ac + bd)}{a^2 + b^2} + \frac{(ad - bc)}{a^2 + b^2}i
\]
Example 1. Factor the following equation considering complex numbers.

\[ x^2 - 2x + 4 = 0 \]

Ans.

From the quadratic formula, we have

\[ x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2} = 1 \pm \sqrt{3} \, i \]

Therefore
Example 2. Let the two roots of the quadratic equation \( ax^2 + bx + c = 0 \) be \( \alpha \) and \( \beta \). Here we admit the case for complex roots. Prove the following relationships.

\[ \alpha + \beta = -\frac{b}{a}, \quad \alpha \beta = \frac{c}{a} \]

Ans.

Using the quadratic formula, we have

\[
\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}
\]

\[
\alpha \beta = \left( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left( \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}
\]
Exercise 1. Simplify the following expressions.

(1) \( (\sqrt{3} + \sqrt{-2})(\sqrt{2} - \sqrt{-3}) \)  
(2) \( 1 + i + i^2 + \ldots + i^{10} \)  
(3) \( \frac{2 + i}{2 - i} + \frac{3 - i}{3 + i} \)

Ans.

Pause the video and solve the problem.
Exercise 1. Simplify the following expressions.

(1) \((\sqrt{3} + \sqrt{-2})(\sqrt{2} - \sqrt{-3})\) \hspace{1cm} (2) \(1 + i + i^2 + \cdots + i^{10}\) \hspace{1cm} (3) \(\frac{2 + i}{2 - i} + \frac{3 - i}{3 + i}\)

Ans.

(1) \((\sqrt{3} + \sqrt{-2})(\sqrt{2} - \sqrt{-3})\) = \((\sqrt{3} + \sqrt{2} i)(\sqrt{2} - \sqrt{3} i)\)

\(= (\sqrt{3}\sqrt{2} + \sqrt{2} \sqrt{3}) + (\sqrt{2}\sqrt{2} - \sqrt{3} \sqrt{3})i = 2\sqrt{6} - i\)

(2) \(1 + i + i^2 + \cdots + i^{10}\) = \((1 + i - 1 - i) + (1 + i - 1 - i) + 1 + i - 1 = i\)

(3) \(\frac{2 + i}{2 - i} + \frac{3 - i}{3 + i}\) = \(\frac{(2 + i)^2}{(2 - i)(2 + i)} + \frac{(3 - i)^2}{(3 + i)(3 - i)}\)

\(= \frac{3 + 4i}{5} + \frac{8 - 6i}{10} = \frac{7 + i}{5}\)
Lesson 7
Complex Numbers

7B
• Complex Plane
• Product of Complex Numbers
• De Moivre’s Formula
A real number can be visualized by a number line.

Complex plane

A complex number can be visualized by a complex plane. Complex plane is established by the real axis and the imaginary axis.

Complex number

\[ z = x + iy \]

Complex plane

displacement in in the \( x \)-axis.

displacement in in the \( y \)-axis.
Polar Coordinate Expression

\((x, y) \quad z = x + iy\)  
: Cartesian coordinate expression

\[x = r \cos \theta, \quad y = r \sin \theta\]

\((r, \theta) \quad z = r(\cos \theta + i \sin \theta)\)  
: Polar coordinate expression

\[r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}\]

\[\theta = \tan^{-1}\left(\frac{y}{x}\right)\]

Modulus

Argument
Product of Complex Numbers

Two complex numbers

\[ z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \]
\[ z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \]

Product

\[ z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\} \]

In the complex plane:

- the modulus of the product is given by the product of two moduli
- the argument of the product is given by the summation of two arguments

Example 4 Prove this product formula.

Ans.

\[ z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \]
\[ = r_1 r_2 \{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\} \]
\[ = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\} \]
Multiplication of “$i$”

In the case $z_1 = z$, $z_2 = i$

$$z = r (\cos \theta + i \sin \theta)$$

$$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

Product

$$zi = r \left\{ \cos \left( \theta + \frac{\pi}{2} \right) + i \sin \left( \theta + \frac{\pi}{2} \right) \right\}$$

**Multiplying “$i$” to $z$ means the rotation of $z$ by $90^\circ$ without changing its modulus**

That makes sense!
De Moivre’s Formula

From the product rule, we have

\[ z^2 = r (\cos \theta + i \sin \theta) r (\cos \theta + i \sin \theta) = r^2 (\cos 2\theta + i \sin 2\theta) \]

Similarly,

\[ z^3 = r^2 (\cos 2\theta + i \sin 2\theta) r (\cos \theta + i \sin \theta) = r^3 (\cos 3\theta + i \sin 3\theta) \]

\[ \cdots \cdots \cdots \]

This suggests the following formula

\[ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \]

- This connects complex numbers and trigonometry
- This produces formulae for \( \cos n\theta \) and \( \sin n\theta \). (Compare after expanding the left hand side.)
Example 5. Find the following value. \((1 + i)^8\)

**Ans.**

From the position in the complex plane

\[ 1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \]

From De Moivre’s Formula

\[
(1 + i)^8 = \sqrt{2}^8 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^8 = 16 \left( \cos \frac{8\pi}{4} + i \sin \frac{8\pi}{4} \right) \\
= 16(\cos 2\pi + i \sin 2\pi) = 16
\]
Exercise 2. Find the following value. \( (\sqrt{3} + i)^8 \)

Ans.

Pause the video and solve the problem.
Exercise 2. Find the following value. \((\sqrt{3} + i)^8\)

Ans.

From the position in the complex plane

\[
\sqrt{3} + i = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)
\]

From De Moivre’s Formula

\[
\left(\sqrt{3} + i\right)^8 = 2^8 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^8 = 256 \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)
\]

\[
= 256 \left( -\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) = -128 - 128\sqrt{3}i
\]
Exercise 3. Find the coefficients A, B, C in the following equalities.

\[ \cos 5\theta = A \cos^5 \theta + B \cos^3 \theta + C \cos \theta \]

[Note] Utilize the expansion

\[ (x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \]

Ans.

Pause the video and solve the problem.
**Exercise 3.** Find the coefficients $A$, $B$, $C$ in the following equalities.

\[
\cos 5\theta = A \cos^5 \theta + B \cos^3 \theta + C \cos \theta
\]

[Note] Utilize the expansion

\[
(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + x^5
\]

**Ans.** From De Moivre’s Formula \((\cos \theta + i \sin \theta)^5 = (\cos 5\theta + i \sin 5\theta)\)

From the given expansion

\[
(\cos \theta + i \sin \theta)^5 = (\cos \theta)^5 + 5(\cos \theta)^4(i \sin \theta) + 10(\cos \theta)^3(i \sin \theta)^2
\]
\[
+ 10(\cos \theta)^2(i \sin \theta)^3 + 5(\cos \theta)(i \sin \theta)^4 + (i \sin \theta)^5
\]
\[
= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta
\]
\[
+ i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta)
\]

From comparing the real parts

\[
\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta
\]
\[
= \cos^5 \theta - 10 \cos^3 \theta(1 - \cos^2 \theta) + 5 \cos \theta(1 - \cos^2 \theta)^2
\]
\[
= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta
\]

\[
\therefore A = 16, \quad B = -20, \quad C = 5
\]