Chapter 8

The six-term exact sequence

By combining the various results obtained in the previous section we obtain the so-called six-term exact sequence in K-theory. In fact, we already know five of the six maps of this sequence. The last map is called *the exponential map* and is constructed from the Bott map composed with the index map δ_2. With this six-term exact sequence, it is possible to compute the K-theory of several C^*-algebras.

8.1 The exponential map and the six-term exact sequence

For any short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

we define the *exponential map* $\delta_0 : K_0(\mathcal{Q}) \to K_1(\mathcal{J})$ by the composition of the maps

$$K_0(\mathcal{Q}) \xrightarrow{\beta_{\mathcal{Q}}} K_2(\mathcal{Q}) \xrightarrow{\delta_2} K_1(\mathcal{J}),$$

where δ_2 has been defined in the diagram (7.6). In other words if $\bar{\delta}_1$ denotes the index map associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0,$$

then

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\delta_0"] \arrow[d, "\beta_{\mathcal{Q}}"] & K_1(\mathcal{J}) \arrow[d, "\theta_{\mathcal{J}}"] \\ K_1\big(S(\mathcal{Q})\big) \arrow[r, "\bar{\delta}_1"] & K_0\big(S(\mathcal{J})\big) \end{tikzcd} \tag{8.1}$$

is a commutative diagram.

Theorem 8.1.1 (The six-term exact sequence)**.** *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

gives rise to the six-term exact sequence

$$\begin{tikzcd}
K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\
K_0(\mathcal{Q}) \arrow[u, "\delta_0"] & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"]
\end{tikzcd} .$$

Proof. By Proposition 6.3.3 it only remains to show that this sequence is exact at $K_0(\mathcal{Q})$ and at $K_1(\mathcal{J})$. To see it, consider the diagram

$$\begin{tikzcd}
K_2(\mathcal{C}) \arrow[r, "K_2(\psi)"] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] & K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \\
K_0(\mathcal{C}) \arrow[u, "\beta_{\mathcal{C}}", "\cong"'] \arrow[r, "K_0(\psi)"'] & K_0(\mathcal{Q}) \arrow[u, "\beta_{\mathcal{Q}}", "\cong"'] \arrow[r, "\delta_0"'] & K_1(\mathcal{J}) \arrow[u, "\mathrm{id}"'] \arrow[r, "K_1(\varphi)"'] & K_1(\mathcal{C}) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

which is commutative: The left-hand square commutes by naturality of the Bott map, see diagram (7.10), and the center square commutes by the definition of the exponential map. The top row is exact by Proposition 7.2.3, from which one infers that the bottom row is exact as well. □

8.2 An explicit description of the exponential map

The exponential map is composed of two natural maps, and therefore is natural as explained in the following statement:

Proposition 8.2.1. *The exponential map δ_0 is natural in the following sense: Given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0
\end{tikzcd}$$

with two short exact sequences of C^*-algebras and with three $*$-homomorphisms, the diagram

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\delta_0"] \arrow[d, "K_0(\beta)"'] & K_1(\mathcal{J}) \arrow[d, "K_1(\gamma)"] \\ K_0(\mathcal{Q}') \arrow[r, "\delta_0'"] & K_1(\mathcal{J}') \end{tikzcd} \quad . \tag{8.2}$$

is commutative, where δ_0 and δ_0' are the associated exponential maps.

Proof. The diagram (8.2) can be decomposed into two commuting squares

$$\begin{tikzcd} K_0(\mathcal{Q}) \arrow[r, "\beta_{\mathcal{Q}}"] \arrow[d, "K_0(\beta)"] & K_2(\mathcal{Q}) \arrow[r, "\delta_2"] \arrow[d, "K_2(\beta)"] & K_1(\mathcal{J}) \arrow[d, "K_1(\gamma)"] \\ K_0(\mathcal{Q}') \arrow[r, "\beta_{\mathcal{Q}'}"'] & K_2(\mathcal{Q}') \arrow[r, "\delta_2'"'] & K_1(\mathcal{J}') \end{tikzcd} \quad ,$$

as seen in diagrams (7.10) and (7.8). □

The following proposition is somewhat technical but it provides an explicit description of the exponential map and justifies its name.

Proposition 8.2.2. *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

be a short exact sequence of C^*-algebras, and let $\delta_0 : K_0(\mathcal{Q}) \to K_1(\mathcal{J})$ be its associated exponential map. Let g be an element of $K_0(\mathcal{Q})$. Then $\delta_0(g)$ can be computed as follows.

(i) Let $p \in \mathcal{P}_n(\widetilde{\mathcal{Q}})$ such that $g = [p]_0 - [s(p)]_0$, and let a be a self-adjoint element in $M_n(\widetilde{\mathcal{C}})$ for which $\tilde{\psi}(a) = p$. Then $\tilde{\varphi}(u) = \exp(2\pi i a)$ for precisely one element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$, and $\delta_0(g) = -[u]_1$.

(ii) Suppose that $\mathcal{C}$ is unital, in which case also $\mathcal{Q}$ is unital and ψ is unit preserving. Let $\bar{\varphi} : \widetilde{\mathcal{J}} \to \mathcal{C}$ be given by $\bar{\varphi}(x + \alpha\mathbf{1}) = \varphi(x) + \alpha\mathbf{1}_{\mathcal{C}}$ for any $x \in \mathcal{J}$ and $\alpha \in \mathbb{C}$. Suppose that $g = [p]_0$ for some $p \in \mathcal{P}_n(\mathcal{Q})$, and let a be a self-adjoint element in $M_n(\mathcal{C})$ such that $\psi(a) = p$. Then $\bar{\varphi}(u) = \exp(2\pi i a)$ for precisely one element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$, and $\delta_0(g) = -[u]_1$.

By the standard picture of K_0 provided in Proposition 4.2.1 we can find a projection p as in (i) for any element $g \in K_0(\mathcal{Q})$. In the unital case, any element of $K_0(\mathcal{Q})$ can be described as the difference of two elements of $\mathcal{P}_n(\mathcal{Q})$, as shown in Proposition 3.2.4. In both cases the existence of a self-adjoint lift is provided by Proposition 2.3.1.

Proof. (ii) We assume in this proof that $\mathcal{Q} \neq \{0\}$, and hence the map $\bar{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\mathcal{C})$ is injective for any $n \in \mathbb{N}^*$. The image of $\bar{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\mathcal{C})$ consists of those elements $x \in M_n(\mathcal{C})$ such that $\psi(x) \in M_n(\mathbb{C}\mathbf{1}) \subset M_n(\mathcal{Q})$. Then, since

$$\psi\big(\exp(2\pi i a)\big) = \exp\big(2\pi i \psi(a)\big) = \exp(2\pi i p) = \mathbf{1}_n$$

there exists a unique element $u \in M_n(\widetilde{\mathcal{J}})$ such that $\bar{\varphi}(u) = \exp(2\pi i a)$, and since $\bar{\varphi}(u)$ is unitary, one concludes that $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$. By (8.1) we must show that

$$(\bar{\delta}_1 \circ \beta_{\mathcal{Q}})([p]_0) = \theta_{\mathcal{J}}([u^*]_1), \tag{8.3}$$

where $\bar{\delta}_1 : K_1\big(S(\mathcal{Q})\big) \to K_0\big(S(\mathcal{J})\big)$ denotes the index map associated with the short exact sequence

$$0 \longrightarrow S(\mathcal{J}) \xrightarrow{S(\varphi)} S(\mathcal{C}) \xrightarrow{S(\psi)} S(\mathcal{Q}) \longrightarrow 0. \tag{8.4}$$

Note that we shall here use the picture $S(\mathcal{Q}) = C_0\big((0,1);\mathcal{Q}\big)$, for which $M_k\big(\widetilde{S(\mathcal{Q})}\big)$ is identified with the set of all continuous functions $f : [0,1] \to M_k(\mathcal{Q})$ where $f(0) = f(1) \in M_k(\mathbb{C}\mathbf{1}) \subset M_n(\mathcal{Q})$.

In the setting just mentioned, the projection loop $f_p \in \mathcal{U}_n\big(\widetilde{S(\mathcal{Q})}\big)$ associated with the projection $p \in \mathcal{P}_n(\mathcal{Q})$ is given for any $t \in [0,1]$ by

$$f_p(t) = e^{2\pi i t}p + (\mathbf{1}_n - p) = e^{2\pi i t p}.$$

By Lemma 6.1.1 there exists $v \in \mathcal{U}_{2n}\big(\widetilde{S(\mathcal{C})}\big)$ such that $\widetilde{S(\psi)}(v) = \operatorname{diag}(f_p, f_p^*)$. By using a similar identification, one infers that $v : [0,1] \to \mathcal{U}_{2n}(\mathcal{C})$ is a continuous map with $v(0) = v(1)$ belonging to $\mathcal{U}_{2n}(\mathbb{C}\mathbf{1}) \subset \mathcal{U}_{2n}(\mathcal{C})$, and

$$\psi\big(v(t)\big) = \begin{pmatrix} f_p(t) & 0 \\ 0 & f_p(t)^* \end{pmatrix}, \quad \forall t \in [0,1].$$

As $f_p(0) = f_p(1) = \mathbf{1}_n$ one infers that $v(0) = v(1) = \mathbf{1}_{2n}$.

Now, since a is a self-adjoint lift for p in $M_n(\mathcal{C})$, let us set $z(t) = \exp(2\pi i t a)$ for any $t \in [0,1]$. Then $z(t)$ belongs to $\mathcal{U}_n(\mathcal{C})$, the map $t \mapsto z(t)$ is continuous, and $\psi\big(z(t)\big) = f_p(t)$. Hence one gets

$$\psi\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t)\end{pmatrix}\right) = \mathbf{1}_{2n} \quad \text{and} \quad s\left(v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t)\end{pmatrix}\right) = \mathbf{1}_{2n}.$$

It follows that we can find $w(t)$ in $\mathcal{U}_{2n}(\widetilde{\mathcal{J}})$ with

$$\bar{\varphi}\big(w(t)\big) = v(t)\begin{pmatrix} z(t)^* & 0 \\ 0 & z(t)\end{pmatrix} \quad \text{and} \quad s\big(w(t)\big) = \mathbf{1}_{2n}.$$

Now, $t \mapsto w(t)$ is continuous because $\bar{\varphi}$ is isometric, $w(0) = \mathbf{1}_{2n}$, and

$$\bar{\varphi}\big(w(1)\big) = \begin{pmatrix} z(1)^* & 0 \\ 0 & z(1)\end{pmatrix} = \bar{\varphi}\begin{pmatrix} u^* & 0 \\ 0 & u\end{pmatrix},$$

which shows that $w(1) = \mathrm{diag}(u^*, u)$. By considering the short exact sequence

$$0 \longrightarrow S(\widetilde{\mathcal{J}}) \xrightarrow{\iota} C(\widetilde{\mathcal{J}}) \xrightarrow{\pi(1)} \widetilde{\mathcal{J}} \longrightarrow 0 \tag{8.5}$$

where $\pi(1)$ means the evaluation at the value 1, one infers from Theorem 7.1.3 applied to (8.5) with the unitary element $w \in \mathcal{U}_{2n}\big(\widetilde{C(\widetilde{\mathcal{J}})}\big)$ and the projection $w\left(\begin{smallmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{smallmatrix}\right)w^* \in \mathcal{P}_{2n}\big(\widetilde{S(\widetilde{\mathcal{J}})}\big)$ that

$$\theta_{\mathcal{J}}([u^*]_1) = \left[w\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w^*\right]_0 - \left[\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}\right]_0. \tag{8.6}$$

This corresponds to the r.h.s. of (8.3).

For the l.h.s. of (8.3), recall first that $\beta_{\mathcal{Q}}([p]_0) = [f_p]_1 \in K_1\big(S(\mathcal{Q})\big)$. Note that we also have

$$\bar{\varphi}\left(w(t)\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w(t)^*\right) = v(t)\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}v(t)^*,$$

which implies that

$$\widetilde{S(\varphi)}\left(w\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}w^*\right) = v\begin{pmatrix}\mathbf{1}_n & 0\\ 0 & 0\end{pmatrix}v^*. \tag{8.7}$$

The unitary element v was chosen such that $\widetilde{S(\psi)}(v) = \mathrm{diag}(f_p, f_p^*)$, and so we get from equations (8.7) and from the definition of the index map for the short exact sequence (8.4) that $\bar{\delta}_1([f_p]_1)$ is also equal to the r.h.s. of (8.6). This fact proves (8.3), as expected.

(i) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\iota_{\mathcal{C}}"] & \mathcal{Q} \arrow[r] \arrow[d, "\iota_{\mathcal{Q}}"] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi'"] & \widetilde{\mathcal{C}} \arrow[r, "\psi'"] & \widetilde{\mathcal{Q}} \arrow[r] & 0
\end{tikzcd}$$

where $\varphi' = \iota_{\mathcal{C}} \circ \varphi$ and $\psi' = \tilde{\psi}$. Let δ_0' be the exponential map associated with the short exact sequence in its lower row. By naturality of the exponential map one gets

$$\begin{aligned}
\delta_0\big([p]_0 - [s(p)]_0\big) &= \big(\delta_0' \circ K_0(\iota_{\mathcal{Q}})\big)([p]_0 - [s(p)]_0) \\
&= \delta_0'\big([p]_0 - [s(p)]_0\big) = \delta_0'\big([p]_0\big) - \delta_0'\big([s(p)]_0\big).
\end{aligned}$$

The maps $\tilde{\varphi} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{C}}$ and $\overline{\varphi'} : \widetilde{\mathcal{J}} \to \widetilde{\mathcal{C}}$ coincide. It follows from *(ii)* that there is an element $u \in \mathcal{U}_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(u) = \overline{\varphi'}(u) = \exp(2\pi i a)$ and that $\delta_0'([p]_0) = -[u]_1$. Since $s(p) = \psi'\big(s(p)\big)$, with $s(p)$ viewed as a scalar projection belonging to $M_n(\widetilde{\mathcal{Q}})$ as well as to $M_n(\widetilde{\mathcal{C}})$, it follows that $[s(p)]_0$ belongs to the image of $K_0(\psi')$ and hence to the kernel of δ_0'. This completes the proof. □