Chapter 6

The index map

In this chapter, we introduce the index map associated with the short exact sequence

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0 \tag{6.1}$$

of C^*-algebras. This map is a group homomorphism $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ that gives rise to an exact sequence

$$\begin{tikzcd} K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\ K_0(\mathcal{Q}) & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"] \end{tikzcd} \tag{6.2}$$

The index map generalizes the classical Fredholm index of Fredholm operators on a Hilbert space.

6.1 Definition of the index map

Before introducing the index map, two preliminary lemmas are necessary.

Lemma 6.1.1. *Consider the short exact sequence* (6.1) *and let* $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$.

(i) There exist $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ *and* $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ *such that*

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad s(p) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix},$$

(ii) If v *and* p *are as in* (i) *and if* $w \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ *and* $q \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ *satisfy*

$$\tilde{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(q) = w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*,$$

then $s(q) = \mathrm{diag}(\mathbf{1}_n, 0_n)$ *and* $p \sim_u q$ *in* $\mathcal{P}_{2n}(\widetilde{\mathcal{J}})$.

Proof. (i) Since the unital $*$-homomorphism $\tilde{\psi} : M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{Q}})$ is surjective, it follows from Lemma 2.1.7.(ii) that there exists $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ such that $\tilde{\psi}(v) = \mathrm{diag}(u, u^*)$. By a direct computation one then gets

$$\tilde{\psi}\,(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*) = \mathrm{diag}(\mathbf{1}_n, 0_n)$$

which implies in particular that $\tilde{\psi}\,(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*)$ is a scalar element of $\widetilde{\mathcal{Q}}$. One then infers from Lemma 4.3.1 that $\tilde{\varphi}$ is injective, and that there exists an element $p \in M_{2n}(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(p) = v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*$. Note that since $v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*$ is a projection and $\tilde{\varphi}$ is injective, then p is a projection as well. Finally, since

$$\tilde{\psi}\big(\tilde{\varphi}(p)\big) = \tilde{\psi}\,(v\,\mathrm{diag}(\mathbf{1}_n, 0_n)\,v^*) = \mathrm{diag}(\mathbf{1}_n, 0_n),$$

one infers that $s(p) = \mathrm{diag}(\mathbf{1}_n, 0_n)$.

(ii) The same arguments as above show that $s(q) = \mathrm{diag}(\mathbf{1}_n, 0_n)$. Note also that $\tilde{\psi}(wv^*) = \mathbf{1}_{2n}$. Again by Lemma 4.3.1 one infers that there exists $z \in M_{2n}(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(z) = wv^*$. Note also that because of the injectivity of $\tilde{\varphi}$, z is necessarily unitary. Finally, since

$$\tilde{\varphi}(zpz^*) = wv^*\tilde{\varphi}(p)vw^* = w\,\mathrm{diag}(\mathbf{1}_n, 0_n)w^* = \tilde{\varphi}(q),$$

one deduces that $q = zpz^*$, which means that $p \sim_u q$ in $\mathcal{P}_{2n}(\widetilde{\mathcal{J}})$, as claimed. □

Based on these results, let us define $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{Q}}) \to K_0(\mathcal{J})$ by $\nu(u) = [p]_0 - [s(p)]_0$ for any $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$, where $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ is the one mentioned in the point (i) of the previous Lemma. Note that this map is well-defined because of the point (ii) above. In the following lemma, we gather some additional information on this map ν.

Lemma 6.1.2. *The map $\nu : \mathcal{U}_\infty(\widetilde{\mathcal{Q}}) \to K_0(\mathcal{J})$ has the following properties:*

(i) $\nu(u_1 \oplus u_2) = \nu(u_1) + \nu(u_2)$ *for any* $u_1, u_2 \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$,

(ii) $\nu(\mathbf{1}) = 0$,

(iii) If $u_1, u_2 \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ *and* $u_1 \sim_h u_2$, *then* $\nu(u_1) = \nu(u_2)$,

(iv) $\nu\big(\tilde{\psi}(u)\big) = 0$ *for any* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{C}})$,

(v) $\big[K_0(\varphi)\big]\big(\nu(u)\big) = 0$ *for any* $u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$.

Proof. (i) For $j \in \{1, 2\}$, consider $u_j \in \mathcal{U}_{n_j}(\widetilde{\mathcal{Q}})$ and chose $v_j \in \mathcal{U}_{2n_j}(\widetilde{\mathcal{C}})$ and $p_j \in \mathcal{P}_{2n_j}(\widetilde{\mathcal{J}})$ according to Lemma 6.1.1.(i). In particular, one has

$$\tilde{\psi}(v_j) = \begin{pmatrix} u_j & 0 \\ 0 & u_j^* \end{pmatrix}, \quad \tilde{\varphi}(p_j) = v_j \begin{pmatrix} \mathbf{1}_{n_j} & 0 \\ 0 & 0 \end{pmatrix} v_j^*$$

so that $\nu(u_j) = [p_j]_0 - [s(p_j)]_0$. Let us also introduce the elements $y \in \mathcal{U}_{2(n_1+n_2)}(\mathbb{C})$, $v \in \mathcal{U}_{2(n_1+n_2)}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2(n_1+n_2)}(\widetilde{\mathcal{J}})$ by

$$y = \begin{pmatrix} \mathbf{1}_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n_2} \end{pmatrix}, \quad v = y \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} y^*, \quad p = y \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} y^*.$$

It then follows that

$$\tilde{\psi}(v) = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_1^* & 0 \\ 0 & 0 & 0 & u_2^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_{n_1+n_2} & 0 \\ 0 & 0 \end{pmatrix} v^*,$$

which corresponds to the requirements of Lemma 6.1.1.(i) for $\mathrm{diag}(u_1, u_2)$, and therefore

$$\nu(u_1 \oplus u_2) = [p]_0 - [s(p)]_0 = [p_1 \oplus p_2]_0 - [s(p_1 \oplus p_2)]_0 = \nu(u_1) + \nu(u_2)$$

because $p \sim_u p_1 \oplus p_2$.

(iii) Given $u_1 \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ choose $v_1 \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p_1 \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ such that

$$\tilde{\psi}(v_1) = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix}, \quad \tilde{\varphi}(p_1) = v_1 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_1^*.$$

Then $\nu(u_1) = [p_1]_0 - [s(p_1)]_0$. Since $u_1^* u_2 \sim_h \mathbf{1}_n \sim_h u_1 u_2^*$ in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ we can apply Lemma 2.1.7.(iii) and infer that there exist $a, b \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ with $\tilde{\psi}(a) = u_1^* u_2$ and $\tilde{\psi}(b) = u_1 u_2^*$. By setting then $v_2 := v_1 \,\mathrm{diag}(a, b) \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ we obtain that

$$\tilde{\psi}(v_2) = \begin{pmatrix} u_2 & 0 \\ 0 & u_2^* \end{pmatrix}, \quad v_2 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_2^* = v_1 \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v_1^* = \tilde{\varphi}(p_1).$$

Thus, one can choose $p_2 = p_1$ and its satisfies $\tilde{\varphi}(p_2) = v_2 \,\mathrm{diag}(\mathbf{1}_n, 0_n)\, v_2^*$. Finally, by the definition of ν one infers that $\nu(u_2) = [p_1]_0 - [s(p_1)]_0 = \nu(u_1)$.

(iv) For $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ let us set $v = \mathrm{diag}(u, u^*) \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p = \mathrm{diag}(\mathbf{1}_n, 0_n) \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ so that $p = s(p)$. It then follows that

$$\tilde{\psi}(v) = \begin{pmatrix} \tilde{\psi}(u) & 0 \\ 0 & \tilde{\psi}(u^*) \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*,$$

and thus $\nu\big(\tilde{\psi}(u)\big) = [p]_0 - [s(p)]_0 = 0$.

The statement (ii) is then a direct consequence of (iv), and the statement (v) follows from the fact that $\tilde{\varphi}(p)$ is unitarily equivalent to $s\big(\tilde{\varphi}(p)\big)$ in $M_{2n}(\widetilde{\mathcal{C}})$ when p is a projection in $M_{2n}(\widetilde{\mathcal{J}})$ associated with u as in Lemma 6.1.1.(i). $\square$

By the previous lemma, one deduces that the three conditions required in Proposition 5.1.4 are satisfied. Thus it follows from the universal property of K_1 that there exists a unique group homomorphism $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ satisfying $\delta_1([u]_1) = \nu(u)$ for any $u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$.

Definition 6.1.3. *The unique group homomorphism* $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ *which satisfies*

$$\delta_1([u]_1) = \nu(u) \qquad \forall u \in \mathcal{U}_\infty(\widetilde{\mathcal{Q}})$$

is called the index map *associated with the short exact sequence* (6.1)*.*

Let us summarize the main properties of the index map in the following statement:

Proposition 6.1.4 (First standard picture of the index map)**.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

be a short sequence of C^*-algebras, let n be a natural number, and consider $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$, $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ which satisfy

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*.$$

Then

$$\delta_1([u]_1) = [p]_0 - [s(p)]_0.$$

Moreover, one has

(i) $\delta_1 \circ K_1(\psi) = 0$,

(ii) $K_0(\varphi) \circ \delta_1 = 0$.

Proposition 6.1.5 (Naturality of the index map)**.** *Let*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r, "\varphi"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathcal{J}' \arrow[r, "\varphi'"] & \mathcal{C}' \arrow[r, "\psi'"] & \mathcal{Q}' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with two short exact sequences of C^*-algebras, and with α, β, γ three $*$-homomorphisms. Let $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ and $\delta_1' : K_1(\mathcal{Q}') \to K_0(\mathcal{J}')$ be the index map associated with the short exact sequences. Then the following diagram is commutative:

$$\begin{tikzcd} K_1(\mathcal{Q}) \arrow[r, "\delta_1"] \arrow[d, "K_1(\beta)"] & K_0(\mathcal{J}) \arrow[d, "K_0(\gamma)"] \\ K_1(\mathcal{Q}') \arrow[r, "\delta_1'"] & K_0(\mathcal{J}') \end{tikzcd}\ .$$

Proof. Let g be an element in $K_1(\mathcal{Q})$, and let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ with $g = [u]_1$. By Lemma 6.1.1.(i) there exists $v \in \mathcal{U}_{2n}(\widetilde{\mathcal{C}})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ such that

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \tilde{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*.$$

Set $v' := \tilde{\alpha}(v)$ in $\mathcal{U}_{2n}(\widetilde{\mathcal{C}'})$ and $p' := \tilde{\gamma}(p)$ in $\mathcal{P}_{2n}(\widetilde{\mathcal{J}'})$. Then one has

$$\begin{aligned} \tilde{\psi}'(v') &= (\widetilde{\psi' \circ \alpha})(v) = (\widetilde{\beta \circ \psi})(v) = \tilde{\beta}\big(\tilde{\psi}(v)\big) = \begin{pmatrix} \tilde{\beta}(u) & 0 \\ 0 & \tilde{\beta}(u)^* \end{pmatrix}, \\ \tilde{\varphi}'(p') &= (\widetilde{\varphi' \circ \gamma})(p) = (\widetilde{\alpha \circ \varphi})(p) = \tilde{\alpha}(v) \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} \tilde{\alpha}(v)^* = v' \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} (v')^* , \end{aligned}$$

which corresponds to the requirements of Lemma 6.1.1.(i) for $\tilde{\beta}(u)$, Then, from the definition of the index map one has

$$\begin{aligned} \big(\delta_1' \circ K_1(\beta)\big)(g) &= \delta_1'\big([\tilde{\beta}(u)]_1\big) = [p']_0 - [s(p')]_0 \\ &= [\tilde{\gamma}(p)]_0 - \big[s\big(\tilde{\gamma}(p)\big)\big]_0 = K_0(\gamma)\big([p]_0 - [s(p)]_0\big) \\ &= K_0(\gamma)\big(\delta_1([u]_1)\big) = \big(K_0(\gamma) \circ \delta_1\big)(g). \end{aligned}$$

This shows that $\delta_1' \circ K_1(\beta) = K_0(\gamma) \circ \delta_1$. □

6.2 The index map and partial isometries

In this section we provide another picture of the index map, which is more intuitive and more useful in applications. The key point in the construction is the following lemma.

Lemma 6.2.1. *Let $\psi : \mathcal{C} \to \mathcal{Q}$ be a surjective $*$-homomorphism between C^*-algebras, and suppose that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and ψ is unit preserving). Then for each unitary element $u \in \mathcal{Q}$ there exists a partial isometry $v \in M_2(\mathcal{C})$ such that*

$$\psi(v) = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.3}$$

Proof. For any $u \in \mathcal{U}(\mathcal{Q})$, there exists by Proposition 2.3.1.(i) an element $c \in \mathcal{C}$ such that $\psi(c) = u$ and $\|c\| = \|u\| = 1$. We then set

$$v = \begin{pmatrix} c & 0 \\ (\mathbf{1} - c^*c)^{1/2} & 0 \end{pmatrix}$$

and check that $v^*v = \mathrm{diag}(\mathbf{1}, 0)$. By taking Exercise 2.2.3 into account, one infers that vv^* is a projection as well, and that v is a partial isometry. From the equalities $\psi(c) = u$ and $\psi\big((\mathbf{1} - c^*c)^{1/2}\big) = (\mathbf{1} - u^*u)^{1/2} = 0$, one deduces that (6.3) holds. □

Proposition 6.2.2 (Second standard picture of the index map)**.** *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

be a short sequence of C^*-algebras. Let $n \leq m$ be a natural numbers, let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ and let v be a partial isometry in $M_m(\widetilde{\mathcal{C}})$ with

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & 0_{m-n} \end{pmatrix}. \tag{6.4}$$

Then $\mathbf{1}_m - v^*v = \tilde{\varphi}(p)$ and $\mathbf{1}_m - vv^* = \tilde{\varphi}(q)$ for some projections p, q in $\mathcal{P}_m(\widetilde{\mathcal{J}})$, and the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ is given by

$$\delta_1([u]_1) = [p]_0 - [q]_0. \tag{6.5}$$

Before providing the proof observe that Lemma 6.2.1 ensures the existence of a partial isometry v satisfying (6.4). It is then a consequence of the above proposition that the r.h.s. of (6.5) does not depend on the choice of v.

Proof. Let us set $e = \mathbf{1}_m - v^*v$ and $f = \mathbf{1}_m - vv^*$ in $\mathcal{P}_m(\widetilde{\mathcal{C}})$. Then one has $\tilde{\psi}(e) = \tilde{\psi}(f) = \operatorname{diag}(0_n, \mathbf{1}_{m-n})$. Because $\tilde{\psi}(e)$ and $\tilde{\psi}(f)$ are scalar matrices, it follows from Lemma 4.3.1.(*ii*) that there are projections $p, q \in \mathcal{P}_m(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(p) = e$ and $\tilde{\varphi}(q) = f$, and $s(p) = s(q) = \operatorname{diag}(0_n, \mathbf{1}_{m-n})$. Let us then set

$$w := \begin{pmatrix} v & f \\ e & v^* \end{pmatrix}, \quad r := \begin{pmatrix} \mathbf{1}_m - q & 0 \\ 0 & p \end{pmatrix}, \quad z := \begin{pmatrix} \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m-n} \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_{m-n} & 0 & 0 \end{pmatrix}.$$

Then r is a projection in $M_{2m}(\widetilde{\mathcal{J}})$, w is a unitary element of $M_{2m}(\widetilde{\mathcal{C}})$ and z is a self-adjoint unitary matrix in $M_{2m}(\mathbb{C})$. In addition, $zw \in \mathcal{U}_{2m}(\widetilde{\mathcal{C}})$, and one has

$$\tilde{\psi}(zw) = z \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m-n} \\ 0 & 0 & u^* & 0 \\ 0 & \mathbf{1}_{m-n} & 0 & 0 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix},$$

where $u_1 = \operatorname{diag}(u, \mathbf{1}_{m-n})$ in $\mathcal{U}_m(\widetilde{\mathcal{Q}})$. One also observes that

$$zw \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & 0 \end{pmatrix} w^* z^* = z \begin{pmatrix} vv^* & ve \\ ev^* & e \end{pmatrix} z^* = z \begin{pmatrix} \mathbf{1}_m - f & 0 \\ 0 & e \end{pmatrix} z^* = \tilde{\varphi}(zrz^*).$$

Since $zrz \in \mathcal{P}_{2m}(\widetilde{\mathcal{J}})$, it finally follows from the definition of the index map that

$$\begin{aligned} \delta_1([u]_1) = \delta_1([u_1]_1) &= [zrz^*]_0 - [s(zrz^*)]_0 = [r]_0 - [s(r)]_0 \\ &= [\mathbf{1}_m - q]_0 + [p]_0 - [\mathbf{1}_n]_0 - [\mathbf{1}_{m-n}]_0 = [p]_0 - [q]_0, \end{aligned}$$

as desired. □

Note that if $\mathcal{J}$ is an ideal in $\mathcal{C}$ and if φ is the inclusion map, then (6.5) can be rephrased as

$$\delta_1([u]_1) = [\mathbf{1}_m - v^*v]_0 - [\mathbf{1}_m - vv^*]_0, \tag{6.6}$$

where m, n are integers with $m \geq n$, u belongs to $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ and v is a partial isometry in $M_m(\widetilde{\mathcal{C}})$ that lifts $\mathrm{diag}(u, 0_{m-n})$.

Similarly, if $\mathcal{C}$ and $\mathcal{Q}$ are unital C^*-algebras, it would be convenient to have a direct expression for the index map. The following statement deals with such a situation for both pictures of the index map.

Proposition 6.2.3. *Let*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

be a short sequence of C^*-algebras, and suppose that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and ψ is unit preserving). Let $\bar{\varphi} : \widetilde{\mathcal{J}} \to \mathcal{C}$ be the $*$-homomorphism defined by $\bar{\varphi}(a + \alpha \mathbf{1}_{\widetilde{\mathcal{C}}}) = \varphi(a) + \alpha \mathbf{1}_{\mathcal{C}}$ for any $a \in \mathcal{J}$ and $\alpha \in \mathbb{C}$. Let also $u \in \mathcal{U}_n(\mathcal{Q})$.

(i) If $v \in \mathcal{U}_{2n}(\mathcal{C})$ and $p \in \mathcal{P}_{2n}(\widetilde{\mathcal{J}})$ are such that

$$\bar{\varphi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad \psi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix},$$

then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$,

(ii) Let $m \geq n$ be integers and let v be partial isometry in $M_m(\mathcal{C})$ with $\psi(v) = \mathrm{diag}(u, 0_{m-n})$, then $\mathbf{1}_m - v^*v = \bar{\varphi}(p)$ and $\mathbf{1}_m - vv^* = \bar{\varphi}(q)$ for some p, q in $\mathcal{P}_m(\widetilde{\mathcal{J}})$, and $\delta_1([u]_1) = [p]_0 - [q]_0$.

We refer to [RLL00, Prop. 9.2.3] for the proof of the above statement. Let us provide one more version of the previous results when u in $\mathcal{U}_n(\widetilde{\mathcal{Q}})$ or u in $\mathcal{U}_n(\mathcal{Q})$ lifts to a partial isometry in $M_n(\widetilde{\mathcal{Q}})$ or in $M_n(\mathcal{Q})$, respectively, and where for further simplification we assume that $\mathcal{J}$ is an ideal of $\mathcal{C}$.

Proposition 6.2.4. *Let*

$$0 \longrightarrow \mathcal{J} \xhookrightarrow{\iota} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

be a short sequence of C^*-algebras, where $\mathcal{J}$ is an ideal in $\mathcal{C}$ and ι is the inclusion map.

(i) Let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ and let $v \in M_n(\widetilde{\mathcal{C}})$ be a partial isometry such that $\tilde{\psi}(v) = u$. Then $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ are projections in $M_n(\mathcal{J})$, and

$$\delta_1([u]_1) = [\mathbf{1}_n - v^*v]_0 - [\mathbf{1}_n - vv^*]_0. \tag{6.7}$$

(ii) Assume that $\mathcal{C}$ is unital (in which case $\mathcal{Q}$ is unital as well and ψ is unit preserving), and let $u \in \mathcal{U}_n(\mathcal{Q})$ which has a lift to a partial isometry $v \in M_n(\mathcal{C})$. Then $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ are projections in $M_n(\mathcal{J})$, and

$$\delta_1([u]_1) = [\mathbf{1}_n - v^*v]_0 - [\mathbf{1}_n - vv^*]_0.$$

Proof. (i) Since

$$\tilde{\psi}(\mathbf{1}_n - v^*v) = \mathbf{1}_n - u^*u = 0, \qquad \tilde{\psi}(\mathbf{1}_n - vv^*) = \mathbf{1}_n - uu^* = 0, \tag{6.8}$$

we see that $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ belong to $M_n(\mathcal{J})$, and these two elements are projections because v is a partial isometry. The identity (6.7) follows from Proposition 6.2.2 together with (6.6).

(ii) Here $\psi(\mathbf{1}_n - v^*v) = \mathbf{1}_n - u^*u = 0$ and $\psi(\mathbf{1}_n - vv^*) = \mathbf{1}_n - uu^* = 0$, showing that $\mathbf{1}_n - v^*v$ and $\mathbf{1}_n - vv^*$ belong to $M_n(\mathcal{J})$. The statement can then be inferred from Proposition 6.2.3.(ii). □

6.3 An exact sequence of K-groups

In this section, we show that the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

induces the exact sequence (6.2) at the level of K-groups, with $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ the index map introduced in the previous sections. For that purpose and without loss of generality, we shall assume that $\mathcal{J}$ is an ideal in $\mathcal{C}$ and that the map φ is the inclusion map. In this case, $M_n(\widetilde{\mathcal{J}})$ is a unital C^*-subalgebra of $M_n(\widetilde{\mathcal{C}})$ for each $n \in \mathbb{N}^*$.

Lemma 6.3.1. *The kernel of the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$ is contained in the image of the map $K_1(\psi) : K_1(\mathcal{C}) \to K_1(\mathcal{Q})$.*

Proof. Let $g \in K_1(\mathcal{Q})$ such that $\delta_1(g) = 0$, and let $u \in \mathcal{U}_n(\widetilde{\mathcal{Q}})$ such that $g = [u]_1$. By Lemma 6.2.1 there exists a partial isometry $w_1 \in M_{2n}(\widetilde{\mathcal{C}})$ such that

$$\tilde{\psi}(w_1) = \begin{pmatrix} u & 0 \\ 0 & 0_n \end{pmatrix}.$$

Then, by Proposition 6.2.2 one infers that the following equalities hold in $K_0(\mathcal{J})$:

$$0 = \delta_1(g) = \delta_1([u]_1) = [\mathbf{1}_{2n} - w_1^*w_1]_0 - [\mathbf{1}_{2n} - w_1w_1^*]_0 \ .$$

By Proposition 3.2.4, one infers that there exists $k \in \mathbb{N}$ and a partial isometry $w_2 \in M_{2n+k}(\widetilde{\mathcal{J}})$ such that

$$(\mathbf{1}_{2n} - w_1^*w_1) \oplus \mathbf{1}_k = w_2^*w_2, \quad (\mathbf{1}_{2n} - w_1w_1^*) \oplus \mathbf{1}_k = w_2w_2^*.$$

Computing the image of these elements through $\tilde{\psi}$ one gets

$$\tilde{\psi}(w_2^*w_2) = \begin{pmatrix} 0_n & 0 \\ 0 & \mathbf{1}_{n+k} \end{pmatrix} = \tilde{\psi}(w_2w_2^*).$$

In addition, since $w_2 \in M_{2n+k}(\widetilde{\mathcal{J}})$ one deduces from Lemma 4.3.1 that $\tilde{\psi}(w_2)$ is a scalar matrix. As a consequence, one has $\tilde{\psi}(w_2) = \mathrm{diag}(0_n, z)$ for some scalar and unitary matrix $z \in M_{n+k}(\widetilde{\mathcal{Q}})$. Since $\mathcal{U}_{n+k}(\mathbb{C})$ is connected, *cf.* Corollary 2.1.3, one finally deduces that z is homotopic to $\mathbf{1}_{n+k}$ in $\mathcal{U}_{n+k}(\widetilde{\mathcal{Q}})$.

Let us now set $v := \mathrm{diag}(w_1, 0_k) + w_2$. One can observe that $v \in \mathcal{U}_{2n+k}(\widetilde{\mathcal{C}})$, and that

$$\tilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & 0_{n+k} \end{pmatrix} + \begin{pmatrix} 0_n & 0 \\ 0 & z \end{pmatrix} \sim_h \begin{pmatrix} u & 0 \\ 0 & \mathbf{1}_{n+k} \end{pmatrix} \quad \text{in } \mathcal{U}_{2n+k}(\widetilde{\mathcal{Q}}).$$

This proves that

$$g = [u]_1 = [\tilde{\psi}(v)]_1 = K_1(\psi)([v]_1),$$

as desired. □

Lemma 6.3.2. *The kernel of the map $K_0(\varphi) : K_0(\mathcal{J}) \to K_0(\mathcal{C})$ is contained in the image of the index map $\delta_1 : K_1(\mathcal{Q}) \to K_0(\mathcal{J})$.*

Proof. Let $g \in K_0(\mathcal{J})$ with $g \in \mathsf{Ker}\big(K_0(\varphi)\big)$. By Lemma 4.2.2, there exist $n \in \mathbb{N}$, $p \in \mathcal{P}_n(\widetilde{\mathcal{J}})$ and $w \in \mathcal{U}_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $wpw^* = s(p)$.

The element $u_0 := \tilde{\psi}\big(w(\mathbf{1}_n - p)\big)$ is a partial isometry in $M_n(\widetilde{\mathcal{Q}})$ and

$$\mathbf{1}_n - u_0^* u_0 = \tilde{\psi}(p) = \tilde{\psi}\big(s(p)\big) = \mathbf{1}_n - u_0 u_0^*,$$

where Lemma 4.3.1 has been used for the second equality. It follows that u_0 is a partial isometry and is normal, and that

$$u := u_0 + (\mathbf{1}_n - u_0^* u_0)$$

is a unitary element in $M_n(\widetilde{\mathcal{Q}})$. In order to lift $\mathrm{diag}(u, 0_n)$ to a suitable partial isometry v in $M_{2n}(\widetilde{\mathcal{C}})$, let us first observe that $v_1 := \mathrm{diag}\big(w(\mathbf{1}_n - p), s(p)\big)$ in $M_{2n}(\widetilde{\mathcal{C}})$ satisfies $\tilde{\psi}(v_1) = \mathrm{diag}\big(u_0, s(p)\big)$. Let $z \in M_{2n}(\mathbb{C})$ be the self-adjoint unitary matrix given by

$$z := \begin{pmatrix} \mathbf{1}_n - s(p) & s(p) \\ s(p) & \mathbf{1}_n - s(p) \end{pmatrix},$$

and set $v := z v_1 z^*$. Then one has

$$\tilde{\psi}(v) = z\tilde{\psi}(v_1)z^* = z \begin{pmatrix} u_0 & 0 \\ 0 & s(p) \end{pmatrix} z^* = \begin{pmatrix} u & 0 \\ 0 & 0_n \end{pmatrix}.$$

It finally follows from Proposition 6.2.2 that

$$\begin{aligned} \delta_1([u]_1) &= [\mathbf{1}_{2n} - v^* v]_0 - [\mathbf{1}_{2n} - vv^*]_0 = [\mathbf{1}_{2n} - v_1^* v_1]_0 - [\mathbf{1}_{2n} - v_1 v_1^*]_0 \\ &= \left[\begin{pmatrix} p & 0 \\ 0 & \mathbf{1}_n - s(p) \end{pmatrix}\right]_0 - \left[\begin{pmatrix} s(p) & 0 \\ 0 & \mathbf{1}_n - s(p) \end{pmatrix}\right]_0 \\ &= [p]_0 - [s(p)]_0 \\ &= g \end{aligned}$$

in $K_0(\mathcal{J})$, and this proves the statement. □

By combining the contents of Propositions 4.3.2, 5.2.3, and 6.1.4, together with Lemmas 6.3.1 and 6.3.2 one gets the following result:

Proposition 6.3.3. *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0$$

gives rise to the following exact sequence of Abelian groups:

$$\begin{tikzcd} K_1(\mathcal{J}) \arrow[r, "K_1(\varphi)"] & K_1(\mathcal{C}) \arrow[r, "K_1(\psi)"] & K_1(\mathcal{Q}) \arrow[d, "\delta_1"] \\ K_0(\mathcal{Q}) & K_0(\mathcal{C}) \arrow[l, "K_0(\psi)"] & K_0(\mathcal{J}) \arrow[l, "K_0(\varphi)"] \end{tikzcd} .$$

Extension 6.3.4. *Study the classical situation of Fredholm operators and Fredholm index, as presented for example in [RLL00, Sec. 9.4].*