Chapter 4

K_0-group for an arbitrary C^*-algebra

In this chapter, we extend the construction of the K_0-group for a non-unital C^*-algebra, and show that this definition is coherent with the previous one when the algebra has a unit.

4.1 Definition and functoriality of K_0

Definition 4.1.1. *Let $\mathcal{C}$ be a non-unital C^*-algebra, and consider the associated split exact sequence*

$$0 \longrightarrow \mathcal{C} \xhookrightarrow{\iota} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

One defines $K_0(\mathcal{C})$ as the kernel of the homomorphism $K_0(\pi) : K_0(\widetilde{\mathcal{C}}) \to K_0(\mathbb{C})$.

Clearly, $K_0(\mathcal{C})$ is an Abelian group, being a subgroup of the Abelian group $K_0(\widetilde{\mathcal{C}})$. In addition, consider $p \in \mathcal{P}_\infty(\mathcal{C})$ and the equivalence class $[p]_0 \in K_0(\widetilde{\mathcal{C}})$. Since by (3.3) one has

$$K_0(\pi)([p]_0) = [\pi(p)]_0 = 0,$$

it follows that $[p]_0$ belongs to $K_0(\mathcal{C})$. In this way, we obtain a map $[\cdot]_0 : \mathcal{P}_\infty(\mathcal{C}) \to K_0(\mathcal{C})$.

Now, for any C^*-algebra, unital or not, we have a short exact sequence

$$0 \longrightarrow K_0(\mathcal{C}) \longrightarrow K_0(\widetilde{\mathcal{C}}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0. \tag{4.1}$$

Note that the map $K_0(\mathcal{C}) \longrightarrow K_0(\widetilde{\mathcal{C}})$ corresponds to $K_0(\iota)$ when $\mathcal{C}$ is unital while it simply corresponds to the inclusion map when $\mathcal{C}$ is not unital. Note also that in the unital case, it has been proved in Lemma 3.3.5 that (4.1) is indeed a short exact sequence while for the non-unital case, this follows from the definition of $K_0(\mathcal{C})$.

When $\mathcal{C}$ is unital, $K_0(\mathcal{C})$ is isomorphic to its image in $K_0(\widetilde{\mathcal{C}})$ through the map $K_0(\iota)$, and $K_0(\iota)$ maps $[p]_0 \in K_0(\mathcal{C})$ to $[p]_0 \in K_0(\widetilde{\mathcal{C}})$ for any $p \in \mathcal{P}_\infty(\mathcal{C})$. Since the image of

$K_0(\iota)$ is equal to the kernel of $K_0(\pi)$, the identity

$$K_0(\mathcal{C}) = \mathsf{Ker}\big(K_0(\pi)\big)$$

holds, for both unital and non-unital C^*-algebras (with a slight abuse of notation).

Let us now consider a $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between C^*-algebras, and let $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ be the corresponding $*$-homomorphism introduced right after Exercise 1.1.10. The commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{C}} \arrow[r, "\pi_{\mathcal{C}}"] \arrow[d, "\tilde{\varphi}"'] & \mathbb{C} \arrow[d, "\mathrm{id}"] \\
\mathcal{Q} \arrow[r, "\iota_{\mathcal{Q}}"'] & \widetilde{\mathcal{Q}} \arrow[r, "\pi_{\mathcal{Q}}"'] & \mathbb{C}.
\end{tikzcd}$$

induces by functoriality of K_0 for unital C^*-algebras the following commutative diagram:

$$\begin{tikzcd}
K_0(\mathcal{C}) \arrow[r] \arrow[d, "K_0(\varphi)"'] & K_0(\widetilde{\mathcal{C}}) \arrow[r, "K_0(\pi_{\mathcal{C}})"] \arrow[d, "K_0(\tilde{\varphi})"'] & K_0(\mathbb{C}) \arrow[d, "\mathrm{id}_{K_0(\mathbb{C})}"] \\
K_0(\mathcal{Q}) \arrow[r] & K_0(\widetilde{\mathcal{Q}}) \arrow[r, "K_0(\pi_{\mathcal{Q}})"'] & K_0(\mathbb{C})
\end{tikzcd}$$

where $K_0(\varphi)$ corresponds to the restriction to $K_0(\mathcal{C})$ of the group homomorphism $K_0(\tilde{\varphi}) : K_0(\widetilde{\mathcal{C}}) \to K_0(\widetilde{\mathcal{Q}})$. Note that if $\mathcal{C}$ and $\mathcal{Q}$ are unital, then the above group homomorphism $K_0(\varphi)$ corresponds to the one already introduced in Section 3.3. Note also that the equality

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0 \qquad \forall p \in \mathcal{P}_\infty(\mathcal{C})$$

holds, no matter if $\mathcal{C}$ is unital or not.

We can now state in a greater generality the functorial properties of K_0 which have already been discussed in Proposition 3.3.1 for unital C^*-algebras only. The proof of this statement consists in minor modifications of the one already presented in the unital case.

Proposition 4.1.2 (Functoriality of K_0 (general case)). *Let $\mathcal{J}$, $\mathcal{C}$ and $\mathcal{Q}$ be C^*-algebras. Then*

(i) $K_0(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{K_0(\mathcal{C})}$,

(ii) If $\varphi : \mathcal{J} \to \mathcal{C}$ and $\psi : \mathcal{C} \to \mathcal{Q}$ are $*$-homomorphisms, then

$$K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi),$$

(iii) $K_0(\{0\}) = \{0\}$,

(iv) $K_0(0_{\mathcal{C}\to\mathcal{Q}}) = 0_{K_0(\mathcal{C})\to K_0(\mathcal{Q})}$.

Let us now mention that the homotopy invariance of K_0, as already presented in Proposition 3.3.2 for the unital case, also extends to the present more general setting:

Proposition 4.1.3 (Homotopy invariance of K_0 (general case))**.** *Let $\mathcal{C}$ and $\mathcal{Q}$ be C^*-algebras.*

(i) If $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ are homotopic $*$-homomorphisms, then $K_0(\varphi) = K_0(\psi)$,

(ii) If $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent, then $K_0(\mathcal{C})$ is isomorphic to $K_0(\mathcal{Q})$. More specifically, if (3.4) *is a homotopy between $\mathcal{C}$ and $\mathcal{Q}$, then $K_0(\varphi) : K_0(\mathcal{C}) \to K_0(\mathcal{Q})$ and $K_0(\psi) : K_0(\mathcal{Q}) \to K_0(\mathcal{C})$ are isomorphisms, with $K_0(\varphi)^{-1} = K_0(\psi)$.*

Let us end this section with a construction which will play an important role in the sequel. For any C^*-algebra $\mathcal{C}$ one defines *the cone* $C(\mathcal{C})$ and *the suspension* $S(\mathcal{C})$ by

$$C(\mathcal{C}) := \big\{f \in C([0,1];\mathcal{C}) \mid f(0) = 0\big\}, \tag{4.2}$$

$$S(\mathcal{C}) := \big\{f \in C([0,1];\mathcal{C}) \mid f(0) = f(1) = 0\big\}. \tag{4.3}$$

We have then a short exact sequence

$$0 \longrightarrow S(\mathcal{C}) \overset{\iota}{\hookrightarrow} C(\mathcal{C}) \overset{\pi}{\longrightarrow} \mathcal{C} \longrightarrow 0, \tag{4.4}$$

where ι is the inclusion mapping, and $\pi(f) = f(1)$ for any $f \in C(\mathcal{C})$.

Note that the cone $C(\mathcal{C})$ is homotopy equivalent to the C^*-algebra $\{0\}$. Indeed, for any $t \in [0,1]$ let us define the $*$-homomorphism $\varphi(t) : C(\mathcal{C}) \to C(\mathcal{C})$ by

$$\big[\varphi(t)(f)\big](s) := f(st) \qquad f \in C(\mathcal{C}),\ s \in [0,1].$$

Clearly, the map $[0,1] \ni t \mapsto \big(\varphi(t)\big)(f) \in C(\mathcal{C})$ is continuous, and therefore one has

$$0_{C(\mathcal{C})\to C(\mathcal{C})} = \varphi(0) \sim_h \varphi(1) = \mathrm{id}_{C(\mathcal{C})}.$$

It then easily follows that the C^*-algebra $C(\mathcal{C})$ is homotopy equivalent to $\{0\}$, and then from Proposition 4.1.3.(ii) and Proposition 4.1.2.(iii) that $K_0\big(C(\mathcal{C})\big) = \{0\}$.

4.2 The standard picture of the group K_0

In Proposition 3.2.4, an explicit formulation of the K_0-group for a unital C^*-algebra was provided. In this section, we present a similar picture for general C^*-algebras. This formulation is very convenient whenever explicit computations involving K_0-groups are performed.

Consider an arbitrary C^*-algebra $\mathcal{C}$ and the corresponding split exact sequence

$$0 \longrightarrow \mathcal{C} \stackrel{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

One then defines the *scalar mapping* s by

$$s := \lambda \circ \pi : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}},$$

i.e. $s(a + \alpha\mathbf{1}) = \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$ and with $\mathbf{1}$ the unit of $\widetilde{\mathcal{C}}$. Note that $\pi\big(s(a)\big) = \pi(a)$ for any $a \in \widetilde{\mathcal{C}}$, and that $a - s(a) \in \mathcal{C}$. As usual, we keep the notation s for the induced $*$-homomorphism $M_n(\widetilde{\mathcal{C}}) \to M_n(\widetilde{\mathcal{C}})$. Its image is the subset $M_n(\mathbb{C})$ of $M_n(\widetilde{\mathcal{C}})$ consisting of all matrices with scalar entries. For short, any element $a \in M_n(\mathcal{C})$ or $a \in M_n(\widetilde{\mathcal{C}})$ will be called *a scalar element* if $a = s(a)$. On the other hand, note that $a - s(a)$ belongs to $M_n(\mathcal{C})$ for any $a \in M_n(\widetilde{\mathcal{C}})$.

The scalar mapping is natural in the sense that if $\mathcal{C}$ and $\mathcal{Q}$ are C^*-algebras, and if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, we then get the commutative diagram:

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "s"] \arrow[d, "\tilde{\varphi}"'] & \widetilde{\mathcal{C}} \arrow[d, "\tilde{\varphi}"] \\ \widetilde{\mathcal{Q}} \arrow[r, "s"'] & \widetilde{\mathcal{Q}}. \end{tikzcd} \tag{4.5}$$

The following proposition contains the standard picture of $K_0(\mathcal{C})$:

Proposition 4.2.1. *For any C^*-algebra $\mathcal{C}$ one has*

$$K_0(\mathcal{C}) = \big\{[p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})\big\}. \tag{4.6}$$

Moreover, one has

(i) For any pair of projections $p, q \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ the following conditions are equivalent:

(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,

(b) There exist natural numbers k and ℓ such that $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_\ell$ in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$,

(c) There exist scalar projections r_1 and r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.

(ii) If $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ satisfies $[p]_0 - [s(p)]_0 = 0$, then there exists a natural number m with $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$.

(iii) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism, then

$$K_0(\varphi)\big([p]_0 - [s(p)]_0\big) = \big[\tilde{\varphi}(p)\big]_0 - \big[s\big(\tilde{\varphi}(p)\big)\big]_0$$

for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$.

Proof. To prove that equation (4.6) holds, observe first that for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ it follows from the equality $\pi = \pi \circ s$ that

$$K_0(\pi)\big([p]_0 - [s(p)]_0\big) = [\pi(p)]_0 - [(\pi \circ s)(p)]_0 = 0.$$

From it, one infers that $[p]_0 - [s(p)]_0$ belongs to $K_0(\mathcal{C})$ for any $p \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$.

Conversely, let g be an arbitrary element of $K_0(\mathcal{C})$, and let $n \in \mathbb{N}^*$ and $p', q' \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ be such that $g = [p']_0 - [q']_0$, see (3.2). Then set

$$p := \begin{pmatrix} p' & 0 \\ 0 & \mathbf{1}_n - q' \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Then one has $p, q \in \mathcal{P}_{2n}(\widetilde{\mathcal{C}})$ and

$$[p]_0 - [q]_0 = [p']_0 + [\mathbf{1}_n - q']_0 - [\mathbf{1}_n]_0 = [p']_0 - [q']_0 = g,$$

where we have used that $[\mathbf{1}_n - q']_0 + [q']_0 = [\mathbf{1}_n]_0$. Since $q = s(q)$ and $K_0(\pi)(g) = 0$ we deduce that

$$[s(p)]_0 - [q]_0 = [s(p)]_0 - [s(q)]_0 = K_0(s)(g) = \big(K_0(\lambda) \circ K_0(\pi)\big)(g) = 0.$$

This shows that $g = [p]_0 - [q]_0 = [p]_0 - [s(p)]_0$.

i) Let $p, q \in \mathcal{P}_\infty(\widetilde{\mathcal{C}})$ be given, and suppose that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$. Then $[p \oplus s(q)]_0 = [q \oplus s(p)]_0$, and hence $p \oplus s(q) \sim_s q \oplus s(p)$ in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$, by Proposition 3.2.4.($v$). By the observations made after Definition 3.2.3, there exists $n \in \mathbb{N}$ such that $p \oplus s(q) \oplus \mathbf{1}_n \sim_0 q \oplus s(p) \oplus \mathbf{1}_n$. This shows that ($a$) implies ($c$). To see that ($c$) implies ($b$) note that if r_1 and r_2 are scalar projections in $\mathcal{P}_\infty(\widetilde{\mathcal{C}})$ of dimension k and ℓ, respectively, then $r_1 \sim_0 \mathbf{1}_k$ and $r_2 \sim_0 \mathbf{1}_\ell$ (see Exercise 3.1.4), and hence $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_\ell$.

To see that (b) implies (a) note first that

$$[p \oplus \mathbf{1}_k]_0 - [s(p \oplus \mathbf{1}_k)]_0 = [p]_0 + [\mathbf{1}_k]_0 - [s(p)]_0 - [\mathbf{1}_k]_0 = [p]_0 - [s(p)]_0.$$

Therefore, it is sufficient to show that $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$ when $p \sim_0 q$. Suppose accordingly that $p = v^*v$ and $q = vv^*$ for some partial isometry $v \in M_{m,n}(\widetilde{\mathcal{C}})$. Let $s(v) \in M_{m,n}(\mathbb{C})$, viewed as a subset of $M_{m,n}(\widetilde{\mathcal{C}})$, be the matrix obtained by applying s to each entry of v. Then $s(v)^*s(v) = s(p)$ and $s(v)s(v)^* = s(q)$, and so $s(p) \sim_0 s(q)$. As a consequence, $[p]_0 = [q]_0$ and $[s(p)]_0 = [s(q)]_0$, and this proves that (a) holds.

ii) If $[p]_0 - [s(p)]_0 = 0$, then $p \sim_s s(p)$ by Proposition 3.2.4.(v), and there exists $m \in \mathbb{N}$ such that $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$, see the observations made juste before Proposition 3.2.4. Note that $p \oplus \mathbf{1}_m \sim s(p) \oplus \mathbf{1}_m$ is equivalent to $p \oplus \mathbf{1}_m \sim_0 s(p) \oplus \mathbf{1}_m$ since p and $s(p)$ belong to the same matrix algebra over $\widetilde{\mathcal{C}}$.

iii) By definition one has

$$\begin{aligned} K_0(\varphi)\big([p]_0 - [s(p)]_0\big) &= K_0(\tilde{\varphi})\big([p]_0 - [s(p)]_0\big) \\ &= \big[\tilde{\varphi}(p)\big]_0 - \big[\tilde{\varphi}\big(s(p)\big)\big]_0 = \big[\tilde{\varphi}(p)\big]_0 - \big[s\big(\tilde{\varphi}(p)\big)\big]_0. \end{aligned}$$

□

The following slightly technical statement will be used in the next section. If proof is provided in [RLL00, Lem. 4.2.3].

Lemma 4.2.2. *Let $\mathcal{C}$, $\mathcal{Q}$ be C^*-algebras, and $\varphi : \mathcal{C} \to \mathcal{Q}$ a $*$-homomorphism. Let also g be an element of $K_0(\mathcal{C})$ which belongs to the kernel of $K_0(\varphi)$. Then:*

(i) There exist $n \in \mathbb{N}^*$, $p \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ and a unitary element $u \in M_n(\widetilde{\mathcal{Q}})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s\big(\tilde{\varphi}(p)\big)$.

(ii) If φ is surjective, one can choose $u = \mathbf{1}$ in the point (i).

4.3 Half and split exactness and stability of K_0

Let us start this section with an easy lemma which described what happens when a unit is added to a short exact sequence. The proof of this lemma is left as an exercise.

Lemma 4.3.1. *Consider the short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

and let $n \in \mathbb{N}^*$. Then

(i) The map $\tilde{\varphi} : M_n(\widetilde{\mathcal{J}}) \to M_n(\widetilde{\mathcal{C}})$ is injective,

(ii) An element $a \in M_n(\widetilde{\mathcal{C}})$ belongs to $\mathsf{Ran}(\tilde{\varphi})$ if and only if $\tilde{\psi}(a) = s\big(\tilde{\psi}(a)\big)$, with $s : \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{Q}}$ the scalar mapping.

Proposition 4.3.2 (Half exactness of K_0)**.** *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\psi} \mathcal{Q} \longrightarrow 0,$$

induces an exact sequence of Abelian groups

$$K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \xrightarrow{K_0(\psi)} K_0(\mathcal{Q}),$$

that is $\mathsf{Ran}\,\big(K_0(\varphi)\big) = \mathsf{Ker}\,\big(K_0(\psi)\big)$.

Proof. By functoriality of K_0 one already knows that

$$K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = K_0(0_{\mathcal{J}\to\mathcal{Q}}) = 0_{K_0(\mathcal{J})\to K_0(\mathcal{Q})},$$

which implies that $\mathsf{Ran}\,\big(K_0(\varphi)\big) \subset \mathsf{Ker}\,\big(K_0(\psi)\big)$.

Conversely, assume that $g \in \mathsf{Ker}\,\big(K_0(\psi)\big)$. According to Lemma 4.2.2.(ii) there exist $n \in \mathbb{N}^*$ and $p \in \mathcal{P}_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\psi}(p) = s\big(\tilde{\psi}(p)\big)$. Then by Lemma 4.3.1.$(ii)$ there exists $e \in M_n(\widetilde{\mathcal{J}})$ such that $\tilde{\varphi}(e) = p$. Since by Lemma 4.3.1.(i) the map $\tilde{\varphi}$ is injective, one infers that $e \in \mathcal{P}_n(\widetilde{\mathcal{J}})$. Therefore,

$$g = [\tilde{\varphi}(e)]_0 - \big[s\big(\tilde{\varphi}(e)\big)\big]_0 = \tilde{\varphi}\big([p]_0 - [s(p)]_0\big) = K_0(\varphi)\big([e]_0 - [s(e)]_0\big) \tag{4.7}$$

which thus belongs to $\mathsf{Ran}\,\big(K_0(\varphi)\big)$. Note that the standard picture of $K_0(\mathcal{J})$ has been used in the last equality of (4.7). These two inclusions lead to the statement. ☐

Proposition 4.3.3 (Split exactness of K_0)**.** *Every split exact sequence of C^*-algebras*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\varphi} \mathcal{C} \underset{\lambda}{\overset{\psi}{\rightleftarrows}} \mathcal{Q} \longrightarrow 0$$

induces a split exact sequence of Abelian groups

$$0 \longrightarrow K_0(\mathcal{J}) \xrightarrow{K_0(\varphi)} K_0(\mathcal{C}) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(\mathcal{Q}) \longrightarrow 0.$$

Proof. It follows from Proposition 4.3.2 that the equality $\mathsf{Ran}\big(K_0(\varphi)\big) = \mathsf{Ker}\big(K_0(\psi)\big)$ holds. In addition, from the functoriality of K_0 one infers that

$$\mathrm{id}_{K_0(\mathcal{Q})} = K_0(\mathrm{id}_{\mathcal{Q}}) = K_0(\psi) \circ K_0(\lambda)$$

which implies that $K_0(\psi)$ is surjective and the splitness of the sequence. As a consequence, it only remains to show that $K_0(\varphi)$ is injective.

For the injectivity, let us consider $g \in \mathsf{Ker}\big(K_0(\varphi)\big)$. By Lemma 4.2.2.$(i)$, there exist $n \in \mathbb{N}^*$, $p \in \mathcal{P}_n(\widetilde{\mathcal{J}})$ and a unitary element $u \in M_n(\widetilde{\mathcal{C}})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s\big(\tilde{\varphi}(p)\big)$. Set $v := (\tilde{\lambda} \circ \tilde{\psi})(u^*)u$, and observe that v is a unitary element of $M_n(\tilde{\mathcal{C}})$ and $\tilde{\psi}(v) = \mathbf{1}$. By Lemma 4.3.1.$(ii)$ there exists an element $w \in M_n(\widetilde{\mathcal{J}})$ with $\tilde{\varphi}(w) = v$. In addition, since $\tilde{\varphi}$ is injective, w must be unitary. Then, from the computation (use Lemma 4.3.1.(ii) in the second last equality)

$$\begin{aligned}\tilde{\varphi}(wpw^*) &= v\tilde{\varphi}(p)v^* = (\tilde{\lambda} \circ \tilde{\psi})(u^*)s\big(\tilde{\varphi}(p)\big)(\tilde{\lambda} \circ \tilde{\psi})(u) \\ &= (\tilde{\lambda} \circ \tilde{\psi})\big(u^*s\big(\tilde{\varphi}(p)\big)u\big)\big) = (\tilde{\lambda} \circ \tilde{\psi})\big(\tilde{\varphi}(p)\big) = s\big(\tilde{\varphi}(p)\big) = \tilde{\varphi}\big(s(p)\big)\end{aligned}$$

and by the injectivity of $\tilde{\varphi}$ we conclude that $wpw^* = s(p)$. This shows that $p \sim_u s(p)$ in $M_n(\widetilde{\mathcal{J}})$, and hence that $g = 0$. □

Let us study the behavior of K_0 with respect to direct sum of C^*-algebras.

Proposition 4.3.4. *For any C^*-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ the K_0-groups $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2)$ and $K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$ are isomorphic.*

Proof. For $i \in \{1, 2\}$, recall that $\iota_i : \mathcal{C}_i \to \mathcal{C}_1 \oplus \mathcal{C}_2$ denotes the canonical inclusion $*$-homomorphism (already introduced in Section 1.1) and let us set $\pi_i : \mathcal{C}_1 \oplus \mathcal{C}_2 \to \mathcal{C}_i$ for the projection $*$-homomorphism. The sequence

$$0 \longrightarrow \mathcal{C}_1 \xrightarrow{\iota_1} \mathcal{C}_1 \oplus \mathcal{C}_2 \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} \mathcal{C}_2 \longrightarrow 0,$$

is a split exact short exact sequence of C^*-algebras, and therefore by Proposition 4.3.3 one directly infers that

$$0 \longrightarrow K_0(\mathcal{C}_1) \xrightarrow{K_0(\iota_1)} K_0(\mathcal{C}_1 \oplus \mathcal{C}_2) \underset{K_0(\iota_2)}{\overset{K_0(\pi_2)}{\rightleftarrows}} K_0(\mathcal{C}_2) \longrightarrow 0$$

is a split exact short exact sequence. It then follows by a standard argument (five lemma) that $K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$ is isomorphic to $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2)$, with the isomorphism given by

$$K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2) \ni (g,h) \mapsto K_0(\iota_1)(g) + K_0(\iota_2)(h) \in K_0(\mathcal{C}_1 \oplus \mathcal{C}_2).$$

□

We shall now see on two examples that the functor K_0 is not exact. Note that it would be the case if any short exact sequence of C^*-algebras would be transformed in a short exact sequence at the level of the K_0-groups.

Example 4.3.5. *Consider the exact sequence*

$$0 \longrightarrow C_0\big((0,1)\big) \overset{\iota}{\hookrightarrow} C\big([0,1]\big) \xrightarrow{\psi} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

One deduces from Proposition 4.3.4 and from Example 3.4.1 that $K_0(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{Z}^2$*, and from Example 3.4.3 that* $K_0\big(C\big([0,1]\big)\big) \cong \mathbb{Z}$*. Therefore* $K_0(\psi)$ *can not be surjective.*

Example 4.3.6. *Let* $\mathcal{H}$ *be an infinite dimensional separable Hilbert space, and consider the* C^**-algebra* $\mathcal{B}(\mathcal{H})$*. The* C^**-subalgebra* $\mathcal{K}(\mathcal{H})$ *of compact operators on* $\mathcal{H}$ *is an ideal of* $\mathcal{B}(\mathcal{H})$*, and the quotient algebra* $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ *is called* the Calkin algebra. *Thus we have a short exact sequence*

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \overset{\iota}{\hookrightarrow} \mathcal{B}(\mathcal{H}) \xrightarrow{\psi} \mathcal{Q}(\mathcal{H}) \longrightarrow 0.$$

From Example 3.4.2 one knows that $K_0\big(\mathcal{B}(\mathcal{H})\big) = \{0\}$*. It will be shown later on that* $K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$ *which means that* $K_0(\iota)$ *can not be injective.*

We finally state an important result for the computation of the K_0-groups for C^*-algebras, but refer to [RLL00, Prop. 4.3.8 & 6.4.1] for its proof.

Proposition 4.3.7 (Stability of K_0)**.** *Let* $\mathcal{C}$ *be a* C^**-algebra and let* $n \in \mathbb{N}^*$*. Then* $K_0(\mathcal{C})$ *is isomorphic to* $K_0\big(M_n(\mathcal{C})\big)$*. In addition, for any separable Hilbert space* $\mathcal{H}$ *the following equality holds*

$$K_0\big(\mathcal{C} \otimes \mathcal{K}(\mathcal{H})\big) \cong K_0(\mathcal{C}).$$

Extension 4.3.8. *Work on the notion of ordered Abelian* K_0*-group, as presented for example in [RLL00, Sec. 5.1].*

Extension 4.3.9. *Work on the irrational rotation* C^**-algebra, as introduced in Exercise 5.8 of [RLL00]. This algebra has played an important role in operator algebra, and the literature on the subject is very rich.*

Extension 4.3.10. *Work on the notion of inductive limit of* C^**-algebras, as presented in Chapter 6 of [RLL00], and more precisely in Section 6.2 of this reference.*