Chapter 6

Pseudodifferential calculus

The aim of this chapter is to stress the link between the algebraic framework introduced so far, and the usual pseudodifferential calculus. The first section is related to the content of Chapter 4, and is based on [MPR05, Sec. 1.1]. It consists mainly in an introduction to the Weyl calculus and to the corresponding Moyal product. The subsequent sections are slightly more general and closely related to Chapter 5. The arguments are borrowed from [MPR05, Sec. 3], and the mentioned link is clearly established.

6.1 The Weyl calculus

In section 4.1 we have seen how to define multiplication operators $\varphi(X)$ and convolution operators $\varphi(D)$ on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$. A natural question is how to define a more general operator $f(X,D)$ on $L^2(\mathbb{R}^d)$ for a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$.

This can be seen as the problem of constructing a functional calculus $f \mapsto f(X,D)$ for the family $X_1,\dots,X_d, D_1,\dots,D_d$ of $2d$ self-adjoint, non-commuting operators. One also would like to define a multiplication $(f,g) \mapsto f \circ g$ satisfying $(f \circ g)(X,D) = f(X,D)g(X,D)$ as well as an involution $f \to f^\circ$ leading to $f^\circ(X,D) = f(X,D)^*$. The deviation of $\circ$ from pointwise multiplication is imputable to the fact that X and D do not commute.

The solution of these problems is called *the Weyl calculus*, or simply *the pseudodifferential calculus*. In order to define it, let us set $\Xi := \mathbb{R}^d \times \hat{\mathbb{R}}^d$, which corresponds to the direct product of a locally compact abelian group G and of its dual group $\hat{G}$. Elements of Ξ will be denoted by $\mathsf{x} = (x,\xi)$, $\mathsf{y} = (y,\eta)$ and $\mathsf{z} = (z,\zeta)$. We also set

$$\sigma(\mathsf{x},\mathsf{y}) := \sigma\big((x,\xi),(y,\eta)\big) = y\cdot\xi - x\cdot\eta$$

for the standard *symplectic form* on Ξ. The prescription for $f(X,D) \equiv \mathfrak{Op}(f)$ with $f : \Xi \to \mathbb{C}$ is then defined for $u \in \mathcal{H}$ and $x \in \mathbb{R}^d$ by

$$[\mathfrak{Op}(f)u](x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2},\eta\right) u(y)\,\mathrm{d}y\,\mathrm{d}\eta, \tag{6.1.1}$$

the involution is $f^\circ(\mathsf{x}) := \overline{f(\mathsf{x})}$ and the multiplication (called *the Moyal product*) is

$$(f \circ g)(\mathsf{x}) := \frac{4^d}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})} f(\mathsf{y})\, g(\mathsf{z})\, \mathrm{d}\mathsf{y}\, \mathrm{d}\mathsf{z}. \tag{6.1.2}$$

Obviously, these formulas must be taken with some care: for many symbols f and g they need a suitable reinterpretation. Also, the normalization factors should always be checked once again, since they mainly depend on the conventions of each author.

Exercise 6.1.1. *Check that if $f(x,\xi) = f(\xi)$ (f is independent of x), then $\mathfrak{Op}(f) = f(D)$, while if $f(x,\xi) = f(x)$ (f is independent of ξ), then $\mathfrak{Op}(f) = f(X)$.*

Beside the encouraging results contained in the previous exercise, let us try to show where all the above formulas come from. We consider the strongly continuous unitary maps $\mathbb{R}^d \ni x \mapsto U_x \in \mathscr{U}(\mathcal{H})$ and $\hat{\mathbb{R}}^d \ni \xi \mapsto V_\xi := e^{-iX\cdot\xi} \in \mathscr{U}(\mathcal{H})$, acting on $\mathcal{H}$ as

$$[U_x u](y) = u(y+x) \quad \text{and} \quad [V_\xi u](y) = e^{-iy\cdot\xi}\, u(y), \qquad u \in \mathcal{H},\ y \in \mathbb{R}^d.$$

These operators satisfy the *Weyl form of the canonical commutation relations*

$$U_x V_\xi = e^{-ix\cdot\xi}\, V_\xi U_x, \qquad x \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d, \tag{6.1.3}$$

as well as the identities $U_x U_{x'} = U_{x'} U_x$ and $V_\xi V_{\xi'} = V_{\xi'} V_\xi$ for $x, x' \in \mathbb{R}^d$ and $\xi, \xi' \in \hat{\mathbb{R}}^d$. These can be considered as a reformulation of the content of Exercise 4.1.3 in terms of bounded operators.

A convenient way to condense the maps U and V in a single one is to define *the Schrödinger Weyl system* $\{W(x,\xi) \mid x \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d\}$ by

$$W(\mathsf{x}) \equiv W(x,\xi) := e^{\frac{i}{2}x\cdot\xi}\, U_x V_\xi = e^{-\frac{i}{2}x\cdot\xi}\, V_\xi U_x, \tag{6.1.4}$$

which satisfies the relation $W(\mathsf{x})W(\mathsf{y}) = e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\, W(\mathsf{x}+\mathsf{y})$ for any $\mathsf{x},\mathsf{y} \in \Xi$. This equality encodes all the commutation relations between the basic operators X and D. Explicitly, the action of W on $u \in \mathcal{H}$ is given by

$$[W(x,\xi)u](y) = e^{-i\left(\frac{1}{2}x+y\right)\cdot\xi}\, u(y+x), \qquad x, y \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d. \tag{6.1.5}$$

Now, recall that for a family of m commuting self-adjoint operators $S_1, \dots, S_m$ one usually defines a functional calculus by the formula $f(S) := \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \check{f}(t) e^{-it\cdot S} \mathrm{d}t$, where $t \cdot S = t_1 S_1 + \ldots + t_m S_m$ and $\check{f}$ is the inverse Fourier transform of f, see Remark 1.7.13 for a simplified version of this equality. The formula (6.1.1) can be obtained by a similar computation. For that purpose, let us define the *symplectic Fourier transformation* $\mathcal{F}_\Xi :\ \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ by

$$(\mathcal{F}_\Xi f)(\mathsf{x}) := \frac{1}{(2\pi)^d} \int_\Xi e^{i\sigma(\mathsf{x},\mathsf{y})} f(\mathsf{y})\, \mathrm{d}\mathsf{y}.$$

Now, for any function $f:\Xi\to\mathbb{C}$ belonging to the Schwartz space $\mathcal{S}(\Xi)$, we set

$$\mathfrak{Op}(f) := \frac{1}{(2\pi)^d}\int_\Xi (\mathcal{F}_\Xi^{-1} f)(\mathsf{x})\, W(\mathsf{x})\,\mathrm{d}\mathsf{x}. \tag{6.1.6}$$

By using (6.1.5), one gets formula (6.1.1). Then it is easy to verify that the relation $\mathfrak{Op}(f)\mathfrak{Op}(g) = \mathfrak{Op}(f\circ g)$ holds for $f,g\in\mathcal{S}(\Xi)$ if one uses the Moyal product introduced in (6.1.2).

Exercise 6.1.2. *Check that the above statements are correct, and in particular that the normalization factors are suitably chosen.*

6.2 Generalized pseudodifferential algebras

We have introduced in Section 5.4 the standard twisted crossed products $(\mathscr{C},G,\theta,\omega)$, as well as their family of Schrödinger representations $(\mathcal{H},\pi,U^\lambda)$ with $\mathcal{H}=L^2(G)$, defined by pseudo-trivializations of the 2-cocycle ω. We shall now observe that by a partial Fourier transformation, we get from these data a sort of pseudodifferential calculus. More precisely, certain classes of functions on $G\times\hat{G}$ will be organised in C^*-algebras with a natural involution and a product involving ω and generalizing the Moyal product introduced in (6.1.2). The composition between the partial Fourier transformation and the Schrödinger representation will lead to a rule of assigning operators to symbols belonging to these C^*-algebras.

Let us consider the locally compact abelian group G and its dual group $\hat{G}$ endowed with normalized Haar measures in such a way that the Fourier transformations

$$\mathcal{F}_G : L^1(G)\to C_0(\hat{G}),\quad (\mathcal{F}_G b)(\xi) = \int_G \overline{\xi(x)}\, b(x)\,\mathrm{d}x$$

and

$$\overline{\mathcal{F}}_G : L^1(G)\to C_0(\hat{G}),\quad \big(\overline{\mathcal{F}}_G b\big)(\xi) = \int_G \xi(x)\, b(x)\,\mathrm{d}x$$

induce unitary maps from $L^2(G)$ to $L^2(\hat{G})$. The inverses of these maps act on $L^2(\hat{G})\cap L^1(\hat{G})$ as $\big(\overline{\mathcal{F}}_{\hat{G}} c\big)(x) = \int_{\hat{G}} \xi(x)\, c(\xi)\,\mathrm{d}\xi$ and $(\mathcal{F}_{\hat{G}} c)(x) = \int_{\hat{G}} \overline{\xi(x)}\, c(\xi)\,\mathrm{d}\xi$.

Let us now consider the standard twisted C^*-dynamical system $(\mathscr{C},G,\theta,\omega)$. We define the mapping $\mathbf{1}\otimes\overline{\mathcal{F}}_G : L^1(G;\mathscr{C})\to C_0(\hat{G};\mathscr{C})$ by $\big[\big(\mathbf{1}\otimes\overline{\mathcal{F}}_G\big)(f)\big](\xi) = \int_G \xi(x)\, f(x)\,\mathrm{d}x$ (equality in $\mathscr{C}$). We recall that $L^1(G)\odot\mathscr{C}$ is a dense subspace of $L^1(G;\mathscr{C})$ and observe that $\big(\mathbf{1}\otimes\overline{\mathcal{F}}_G\big)(a\otimes b) = a\otimes\big(\overline{\mathcal{F}}_G b\big)$. Let us now also fix an element $\tau\in\mathsf{End}(G)$. We transport all the structure of the Banach *-algebra $(L^1(G;\mathscr{C}), *^\omega_\tau, ^{*^\omega_\tau}, \|\cdot\|_1)$ to the corresponding subset of $C_0(\hat{G};\mathscr{C})$ via $\mathbf{1}\otimes\overline{\mathcal{F}}_G$. The space $\big(\mathbf{1}\otimes\overline{\mathcal{F}}_G\big)L^1(G;\mathscr{C})$ will also be a Banach *-algebra with a composition law $\circ^\omega_\tau$, an involution $^{\circ^\omega_\tau}$ and the norm $\|(\mathbf{1}\otimes\overline{\mathcal{F}}_G^{-1})\cdot\|_1$. Its enveloping C^*-algebra will be denoted by $\mathfrak{C}^\omega_{\mathscr{C},\tau}$. The map $\mathbf{1}\otimes\overline{\mathcal{F}}_G$ extends canonically to an isomorphism between $\mathscr{C}\rtimes^\omega_{\theta,\tau} G$ and $\mathfrak{C}^\omega_{\mathscr{C},\tau}$. We remark that $\big(\mathbf{1}\otimes\overline{\mathcal{F}}_G\big)[L^1(G)\odot\mathscr{C}]$ is already not very explicit, since one has no direct characterization of the space $\overline{\mathcal{F}}_G[L^1(G)]$.

Concerning $\mathfrak{C}^{\omega}_{\mathscr{C},\tau}$, we do not even know if it consists entirely of $\mathscr{C}$-valued distributions on $\hat{G}$ (whenever this makes sense). However, usually one can work efficiently on suitable dense subsets.

We deduce now the explicit form of the composition law and of the involution. Let us simply denote $\mathbf{1} \otimes \overline{\mathcal{F}_G}$ by $\mathfrak{F}$. One gets for any $f, g \in \mathfrak{F}L^1(G;\mathscr{C})$ (be careful with the position of the arguments)

$$\begin{aligned}(f \circ^{\omega}_{\tau} g)(x;\xi) &:= \left(\mathfrak{F}\left[(\mathfrak{F}^{-1}f) *^{\omega}_{\tau} (\mathfrak{F}^{-1}g)\right]\right)(x;\xi)\\ &= \int_G\int_G\int_{\hat{G}}\int_{\hat{G}} \xi(y)\,\overline{\eta(z)}\,\overline{\zeta(y-z)}\; f(x+\tau(z-y);\eta)\cdot\\ &\qquad \cdot\; g(x+(\mathbf{1}-\tau)z;\zeta)\;\omega(x-\tau y; z, y-z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\eta\,\mathrm{d}\zeta\end{aligned}$$

and

$$\begin{aligned}\left(f^{\circ^{\omega}_{\tau}}\right)(x;\xi) &:= \left(\mathfrak{F}\left[\left(\mathfrak{F}^{-1}f\right)^{*^{\omega}_{\tau}}\right]\right)(x;\xi)\\ &= \int_G\int_{\hat{G}} \left[\xi\cdot\eta^{-1}\right](y)\;\omega(x-\tau y; y, -y)^{-1}\;\overline{f(x+(1-2\tau)y;\eta)}\,\mathrm{d}y\,\mathrm{d}\eta.\end{aligned}$$

Both expressions make sense as iterated integrals; under more stringent conditions on f and g, the integrals will be absolutely convergent.

Exercise 6.2.1. *Show that in the special case $\tau = \frac{1}{2}\mathbf{1}$ and $\omega = 1$, the above formulas correspond to the Moyal product $\circ$ and to the involution $^{\circ}$ introduced in Section 6.1.*

The constructions and formulae presented above can be given (with some slight adaptations) for any (abelian) twisted dynamical system. However, since we are considering a standard twisted dynamical system, it means that ω is pseudo-trivial. Thus, for any continuous function $\lambda : G \to C(G;\mathbb{T})$ such that $\delta^1(\lambda) = \omega$, the corresponding Schrödinger covariant representation $(\mathcal{H}, \pi, U^{\lambda})$ gives rise to the Schrödinger representation of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ that we have denoted by $\mathfrak{Rep}^{\lambda}_{\tau}$ in the previous chapter. As a consequence, we get a representation of $\mathfrak{C}^{\omega}_{\mathscr{C},\tau}$ just by composing with $\mathfrak{F}^{-1}$; and this representation will be denoted by $\mathfrak{Op}^{\lambda}_{\tau}$. By simple computations one obtains:

Proposition 6.2.2. *(i) The representation $\mathfrak{Op}^{\lambda}_{\tau} := \mathfrak{Rep}^{\lambda}_{\tau} \circ \mathfrak{F}^{-1} : \mathfrak{C}^{\omega}_{\mathscr{C},\tau} \to \mathscr{B}(\mathcal{H})$ is faithful and acts on $f \in \mathfrak{F}L^1(G;\mathscr{C})$ with $u \in \mathcal{H}$ and $x \in G$ by the formula*

$$\left[\mathfrak{Op}^{\lambda}_{\tau}(f)u\right](x) = \int_G\int_{\hat{G}} \eta(x-y)\,\lambda(x; y-x)\, f\big((1-\tau)x+\tau y;\eta\big)\,u(y)\,\mathrm{d}y\,\mathrm{d}\eta \tag{6.2.1}$$

where the right-hand side is viewed as an iterated integral.

(ii) If $\mu \in C^1\big(G; C(G;\mathbb{T})\big)$ is another 1-cochain, giving a second pseudo-trivialization of the 2-cocycle ω, then $\mu = \delta^0(c)\lambda$ for some $c \in C(G;\mathbb{T})$ and $\mathfrak{Op}^{\lambda}_{\tau}$, $\mathfrak{Op}^{\mu}_{\tau}$ are unitarily equivalent:

$$\pi(c^{-1})\,\mathfrak{Op}^{\lambda}_{\tau}(f)\,\pi(c) = \mathfrak{Op}^{\mu}_{\tau}(f), \quad \forall f \in \mathfrak{C}^{\omega}_{\mathscr{C},\tau}. \tag{6.2.2}$$

Remark 6.2.3. *One can again observe that in the special case* $\tau = \frac{1}{2}\mathbf{1}$ *and with the choice* $\lambda = 1$ *(absence of* 2*-cocycle), the formula provided in* (6.2.1) *corresponds to the expression provided in* (6.1.1)*.*

Let us recall from from Section 5.2 that for different τ's, the C^*-algebras $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \times^{\omega}_{\theta,\tau} G$ are isomorphic, and therefore $\mathfrak{C}^{\omega}_{\theta,\tau}$ and $\mathfrak{C}^{\omega}_{\theta,\tau}$ are also isomorphic. More precisely, recall that $(m_{\tau,\tau'}f)(q;x) = f(q + (\tau' - \tau)x; x)$ for any $x, q \in G$ and $f \in L^1(G;\mathcal{A})$. Note that this isomorphism satisfies then $\mathfrak{Op}^{\lambda}_{\tau'} = \mathfrak{Op}^{\lambda}_{\tau} \circ m_{\tau,\tau'}$ (here $\circ$ is simply the composition) and thus gives the transformation of the τ-symbol of a generalized pseudodifferential operator into its τ'-symbol.

As already mentioned, in the general literature on twisted crossed product C^*-algebras only the special case $\tau = \mathbf{0}$ is considered. However, in order to make the connection with the usual Weyl calculus on the group $\mathbb{R}^d$, the special choice $\tau = \frac{1}{2}\mathbf{1}$ had to be considered, and this is the reason why we have introduced the larger family $\tau \in \mathsf{End}(G)$. We now support the assertion that the choice of the parameter τ is in fact a matter of ordering. Indeed, let us assume that the G-algebra $\mathscr{C}$ is unital, see Definition 5.4.1 for the notion of G-algebra. Then any element $f = 1 \otimes b$ is in $\mathfrak{C}^{\omega}_{\theta,\tau}$ for any $b : \hat{G} \to \mathbb{C}$ with $\mathcal{F}_{\hat{G}} b \in L^1(G)$. In addition, the operator $\mathfrak{Op}^{\lambda}_{\tau}(1 \otimes b)$ does not depend on τ, see formula (6.2.1). We denote it by $\mathfrak{op}^{\lambda}(b)$; its action on $u \in \mathcal{H}$ is given by

$$\left[\mathfrak{op}^{\lambda}(b)u\right](x) = \int_G \lambda(x; y - x)\,[\overline{\mathcal{F}}_{\hat{G}} b](y - x)\, u(y)\,\mathrm{d}y.$$

Finally, by considering then arbitrary element $a \in \mathscr{C}$, simple computations for $\tau = \mathbf{0}$ and $\tau = \mathbf{1}$ show that $\mathfrak{Op}^{\lambda}_{\mathbf{0}}(a \otimes b) = \pi(a)\mathfrak{op}^{\lambda}(b)$ and $\mathfrak{Op}^{\lambda}_{\mathbf{1}}(a \otimes b) = \mathfrak{op}^{\lambda}(b)\pi(a)$, where $\pi(a)$ denotes the multiplication operator by the function a.

Extension 6.2.4. *Let us stress once more that the set of functions for which the above integrals are absolutely convergent can be rather small, and certainly too small for various applications. Several possible extensions are then possible. A first approach would be to deal with multiplier algebras, as sketched in [MPR05, Sec. 3.3]. An approach by duality (but in a less general framework) has been introduced in [MP04]. Alternatively, technics involving oscillatory integrals have been discussed in [LMR10], also in the magnetic framework introduced in the following chapter. All these extensions allow us to consider the expressions introduced in this chapter for a much larger class of symbols.*