Chapter 3

Crossed product C^*-algebras

The aim of this chapter is to present an introduction to the theory of crossed product C^*-algebras. Our main references will be Chapter 7 of [Ped79] and the first chapters of [Wil07]. We shall also heavily rely on [Fol95] for the preliminary sections on locally compact groups, and refer to [Tom87] for a nice introduction to C^*-dynamical systems and crossed product algebras in a restricted setting.

3.1 Locally compact groups

We start with some information on locally compact group. The main reference is [Fol95].

Definition 3.1.1. *A locally compact group is a group G equipped with a locally compact and Hausdorff*[1] *topology with respect to which the group operations are continuous,* i.e. *$G \times G \ni (x, y) \mapsto xy \in G$ is continuous, and $G \ni x \mapsto x^{-1} \in G$ is continuous. The unit of G is denoted by* 1.

Note that we use the multiplicative notation for the group, and therefore the unit is denoted by 1. If the additive notation is used (and this will be the case at some places in the sequel), then the continuity requirements read $G \times G \ni (x, y) \mapsto x + y \in G$ is continuous, and $G \ni x \mapsto -x \in G$ is continuous, and the unit of G is denoted by 0. In the sequel, G will always denote a locally compact group.

If V is a subset of G, we write $V^{-1} := \{x^{-1} \in G \mid x \in V\}$ and say that V is *symmetric* if $V = V^{-1}$. For two subsets V_1, V_2 of G, we write V_1V_2 for $\{xy \in G \mid x \in V_1 \text{ and } y \in V_2\}$. A subgroup of H of G is *normal* if $xHx^{-1} = H$ for all $x \in G$. In particular, if H is a normal subgroup of G, then its quotient G/H is also a locally compact group.

For any bounded map $f : G \to \mathbb{C}$, we define the left and right translates of f by $[L_y f](x) := f(y^{-1}x)$ and $[R_y f](x) := f(xy)$. These definitions make the maps $y \mapsto L_y$ and $y \mapsto R_y$ group homomorphisms. We say that f is *left uniformly continuous*,

[1] This condition is often tacitly assumed in the literature, and we will always assume it implicitly.

resp. *right uniformly continuous*, if $\|L_y f - f\|_\infty \to 0$, resp. $\|R_y f - f\|_\infty \to 0$, as $y \to 1$ in G.

Let us start with a simple result which is well-known for continuous functions on $\mathbb{R}^d$ with compact support. We use the notation $C_c(G)$ for the set of compactly supported continuous complex functions on G.

Lemma 3.1.2. *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

Proof. We give the proof for the right uniform continuity, the argument for the other one is similar. Let $f \in C_c(G)$, $K := \operatorname{supp} f$ and $\varepsilon > 0$. For every $x \in K$, let U_x be a neighbourhood of 1 such that $|f(xy) - f(x)| < \varepsilon/2$ for any $y \in U_x$, and let V_x be a symmetric neighbourhood of 1 such that $V_x V_x \subset U_x$. The sets xV_x define a covering of K, so there exists $x_1, \dots, x_n \in K$ such that $K \subset \cup_{j=1}^n x_j V_{x_j}$. Let us set $V := \cap_{j=1}^n V_{x_j}$ and show that $\sup_{y \in V} \|R_y f - f\|_\infty < \varepsilon$. Indeed, for any $x \in K$ there exists $j \in \{1, \dots, n\}$ such that $x_j^{-1} x \in V_{x_j}$, so that $xy = x_j(x_j^{-1}x)y \in x_j U_{x_j}$ for any $y \in V$. Then, one has

$$|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Similarly, if $xy \in K$, one obtains $|f(xy) - f(x)| < \varepsilon$. If neither $xy \in K$ nor $x \in K$, then $f(xy) - f(x) = 0 - 0 = 0$, and the statement is proved. □

Definition 3.1.3. A left Haar measure*, resp.* a right Haar measure*, on G is a non-zero Radon measure*[2] *μ on G that satisfies $\mu(xV) = \mu(V)$, resp. $\mu(Vx) = \mu(V)$, for every Borel set $V \subset G$ and every $x \in G$.*

For any Radon measure μ and any set V we write $(\tilde{\mu})(V) := \mu(V^{-1})$. From now on, we also denote by $C_c^+(G)$ the subset of compactly supported continuous functions on G which are non-negative.

Lemma 3.1.4. *Let μ be a Radon measure on G.*

(i) μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure,

(ii) μ is a left Haar measure if and only if $\int_G L_y f \,\mathrm{d}\mu = \int_G f \,\mathrm{d}\mu$ for any $f \in C_c^+(G)$ and any $y \in G$.

The proof of this statement is rather simple, see [Fol95, Prop. 2.9]. In view of this result, it is not really relevant to consider differently a left Haar measure or a right Haar measure. We shall follow the more common choice which consists in considering left Haar measures only.

The following statement is of fundamental importance for performing analysis on locally compact groups. We refer to Theorem 2.10 and 2.20 of [Fol95] for its proof, and for various examples of locally compact group with their Haar measure.

[2] A Radon measure is a measure on the algebra of Borel sets of a Hausdorff topological space X that is locally finite and inner regular.

Theorem 3.1.5. *Every locally compact group possesses a left Haar measure, which is unique up to a scaling constant.*

Note that if μ is a Haar measure on G, then $\mu(V) > 0$ for every non-empty open set V, and that $\int_G f \mathrm{d}\mu > 0$ for any $f \in C_c^+(G)$ with $f \neq 0$.

Extension 3.1.6. *It has not been assumed that the locally compact group G is σ-compact ($\Leftrightarrow$ the union of countably many compact subsets). Consequently, the Haar measure is not always σ-finite ($\Leftrightarrow$ G might not a countable union of measurable sets with finite measure). In such a situation, some standard results of analysis which are well-known on $\mathbb{R}^d$ present some complications for their generalization on G, but these problems are manageable, see [Fol95, Sec. 2.3] for details.*

Let us fix a locally compact group G with a left Haar measure μ. We shall denote by $L^p(G, \mathrm{d}\mu)$ the L^p-spaces constructed with this measure. Now, for any $x \in G$ and $V \subset G$, let us define the measure μ_x by $\mu_x(V) := \mu(Vx)$. μ_x is again a left Haar measure, and by Theorem 3.1.5 there exists $\Delta(x) \in (0, \infty)$ such that $\mu_x = \Delta(x)\mu$. Note that the value $\Delta(x)$ is independent of the original choice for the Haar measure μ. The map $\Delta : G \to \mathbb{R}_+$ is called *the modular function of G*. An important result concerning this function is:

Lemma 3.1.7. *The map Δ is a continuous homomorphism from G to the group multiplicative on $\mathbb{R}_+$. Moreover, for any $f \in L^1(G, \mathrm{d}\mu)$ one has $\int_G R_y f \mathrm{d}\mu = \Delta(y^{-1}) \int_G f \mathrm{d}\mu$.*

Proof. For any $x, y \in G$ and $V \subset G$ one has

$$\Delta(xy)\mu(V) = \mu(Vxy) = \Delta(y)\mu(Vx) = \Delta(y)\Delta(x)\mu(V),$$

so that Δ is a homomorphism from G to $\mathbb{R}_+$. For the rest of the proof, we refer to [Fol95, Prop. 2.24]. □

Definition 3.1.8. *A locally compact group G is called* unimodular *if $\Delta = 1$.*

Abelian groups and discrete groups are unimodular, but many others groups are unimodular too.

Lemma 3.1.9. *If K is any compact subgroup of G, then $\Delta|_K = 1$.*

Proof. $\Delta(K)$ is a compact subgroup of $\mathbb{R}_+$, and therefore $\Delta(K) = \{1\}$. □

Corollary 3.1.10. *If G is compact, then G is unimodular.*

Let us now denote by $M(G)$ the space of all complex bounded Radon measures on G, and endow this set with a convolution and an involution: For any $\mu, \nu \in M(G)$ and $f \in C_0(G)$ we define the convolution $\mu * \nu$ by the formula

$$\int_G f(x) \mathrm{d}(\mu * \nu)(x) = \int_G \int_G f(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y)$$

and the involution by the formula $\int_G f(x)\mathrm{d}\mu^*(x) = \int_G \overline{f(x^{-1})\mathrm{d}\mu(x)}$. Endowed with this product and involution, $M(G)$ becomes a B^*-algebra.

The closed self-adjoint ideal $L^1(G)$ consisting of measures which are absolutely continuous with respect to a left Haar measure on G can clearly be identified with the space $L^1(G, \mathrm{d}\mu)$. With this identification one has for any $f, g \in L^1(G)$

$$f * g(x) = \int_G f(y)g(y^{-1}x)\mathrm{d}\mu(y) = \int_G f(xy)g(y^{-1})\mathrm{d}\mu(y)$$

and $f^*(x) = \Delta(x)^{-1}\overline{f(x^{-1})}$. The B^*-algebra $L^1(G)$ is called *the L^1-group algebra of G*. Note that $M(G)$ is always unital, with unit δ_1 (the point mass at 1) while $L^1(G)$ is unital if and only if G is discrete. However, approximate unit exists for $L^1(G)$:

Theorem 3.1.11. *For any locally compact group G, there exists an approximate unit for $L^1(G)$,* i.e. *there exists an increasing net $\{I_j\}_{j\in J} \subset L^1(G)$ with $I_j \geq 0$, $I_j(x^{-1}) = I_j(x)$, and $\int_G I_j(x)\mathrm{d}\mu(x) = 1$, such that $\lim_j \|f * I_j - f\|_1 = 0$.*

Proof. We refer to [Fol95, Prop. 2.42] for a constructive proof. □

Exercise 3.1.12. *We state in this exercise a couple of useful formulas which can be deduced from the definition of the modular function. Let $f \in C_c(G)$ and $x \in G$:*

$$\begin{aligned}
\int_G f(xy)\,\mathrm{d}\mu(y) &= \int_G f(y)\,\mathrm{d}\mu(y),\\
\int_G f(yx)\,\mathrm{d}\mu(y) &= \Delta(x)^{-1}\int_G f(y)\,\mathrm{d}\mu(y),\\
\int_G \Delta(y^{-1})\,f(y^{-1})\,\mathrm{d}\mu(y) &= \int_G f(y)\,\mathrm{d}\mu(y).
\end{aligned}$$

We end this section with some information on representations. In particular, we shall prove a result about the equivalence between unitary representations of the group and non-degenerate representations of the corresponding L^1-group algebra. Note that we use the notation $\mathscr{U}(\mathcal{H})$ for the set of all unitary operators in a Hilbert space $\mathcal{H}$.

Definition 3.1.13. A unitary representation *of G is a pair $(\mathcal{H}, U)$, where $\mathcal{H}$ is a Hilbert space and where $U : G \to \mathscr{U}(\mathcal{H})$ is a homomorphism which is strongly continuous. One usually writes U_x for $U(x) \in \mathscr{U}(\mathcal{H})$.*

Note that on $\mathscr{U}(\mathcal{H})$, weak and strong topologies coincide. Recall also that a representation of a B^*-algebra $\mathscr{C}$ is a pair $(\mathcal{H}, \pi)$ with $\mathcal{H}$ a Hilbert space and $\pi : \mathscr{C} \to \mathscr{B}(\mathcal{H})$ a continuous $*$-homomorphism. This representation is non-degenerate if for any $h \in \mathcal{H}$ there exists $A \in \mathscr{C}$ such that $\pi(A)h \neq 0$.

Proposition 3.1.14. *There are bijective correspondences between the sets of unitary representations of G, representations of $M(G)$ whose restrictions to $L^1(G)$ are non-degenerate, and non-degenerate representations of $L^1(G)$.*

Proof. If $(\mathcal{H}, U)$ is a unitary representation of G, we define for each $\mu \in M(G)$ and each $h, h' \in \mathcal{H}$

$$\langle \pi(\mu)h, h' \rangle := \int_G \langle U_x h, h' \rangle \mathrm{d}\mu(x). \tag{3.1.1}$$

Then $(\mathcal{H}, \pi)$ us a representation of $M(G)$, and by using an approximate unit for $L^1(G)$ we can check that the restriction to $L^1(G)$ is non-degenerate.

Conversely, let $(\mathcal{H}, \pi)$ be a non-degenerate representation of $L^1(G)$ and let $\{I_j\}_{j \in J}$ be an approximate unit for $L^1(G)$. Since elements of the form $\pi(f)h$ with $f \in L^1(G)$ and $h \in \mathcal{H}$ are dense in $\mathcal{H}$, it follows that $\{\pi(I_j)\}$ converges strongly to the operator $\mathbf{1}$. In addition, this representation can be extended to a representation of $L(G)$ (we use the same symbols for this extension) by defining

$$\pi(\mu)(\pi(f)h) = \pi(\mu * f)h \tag{3.1.2}$$

for any $\mu \in M(G)$, $f \in L^1(G)$ and $h \in \mathcal{H}$. Equivalently, one has

$$\pi(\mu)h = s - \lim_j \pi(\mu * I_j)h, \tag{3.1.3}$$

which shows that the extension is unique. The restriction of $M(G)$ to point measures δ_x with $x \in G$, provides then a unitary representation of G whose extension to $L^1(G)$ is precisely the representation $(\mathcal{H}, \pi)$. $\square$

We refer to [Fol95, Sec. 3.2] for more details in the above proof. Note that a unitary representation of G always exists, namely its *left regular representation*: We consider $\mathcal{H} := L^2(G, \mathrm{d}\mu)$ where μ is a Haar measure on G, and set

$$[U_x f](y) = [L_x f](y) = f(x^{-1}y). \tag{3.1.4}$$

By the construction exhibited in the proof of the previous proposition, one also obtains a non-degenerate representation of $L^1(G)$ on $L^2(G, \mathrm{d}\mu)$, whose norm closure in $\mathscr{B}\big(L^2(G, \mathrm{d}\mu)\big)$ is called *the reduced group C^*-algebra*, and is usually denoted by $\mathscr{C}^*_r(G)$. On the other hand, the completion of $L^1(G)$ with the norm

$$\|f\| := \sup\{\|\pi(f)\| \mid (\mathcal{H}, \pi) \text{ is a unitary representation of } G\}$$

is called *the group C^*-algebra $\mathscr{C}^*(G)$*.

Let us now consider a unitary representation $(\mathcal{H}, U)$ of G. If there exists a non-trivial closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $U_x\mathcal{M} \subset \mathcal{M}$ for all $x \in G$, then $\mathcal{M}$ is called *an invariant subspace for U*. In such a case, the restriction $(\mathcal{M}, U|_{\mathcal{M}})$ is a unitary *subrepresentation* of G. If such a subrepresentation exists, the original representation $(\mathcal{H}, U)$ is called *reducible*, and otherwise *irreducible*.

The following statement is important in this context. Its proof is not difficult but requires some preliminary lemmas, see [Fol95, Lem. 3.5].

Theorem 3.1.15 (Schur's Lemma)**.** *A unitary representation $(\mathcal{H}, U)$ of G is irreducible if and only if the set of elements of $\mathscr{B}(\mathcal{H})$ which commute with U_x for all $x \in G$ is reduced to $\mathbb{C}\mathbf{1}$.*

The set mentioned in the previous statement is usually called *the commutant* or *the centralizer* of $(\mathcal{H}, U)$.

Corollary 3.1.16. *If G is abelian, then every irreducible representation of G is one-dimensional.*

Proof. If $(\mathcal{H}, U)$ is a representation of G, then U_x commute will all elements U_y for any $y \in G$. Therefore, U_x belongs to the commutant of $(\mathcal{H}, U)$ for any $x \in G$. If this representation is irreducible, this commutant is equal to $\mathbb{C}\mathbf{1}$, and therefore we have $U_x = c_x\mathbf{1}$, with $c_x \in \mathbb{C}$, for all $x \in G$. Since every one-dimensional subspace of $\mathcal{H}$ is then invariant for U, it follows that $\dim(\mathcal{H}) = 1$. $\square$

Extension 3.1.17. *The notion of* amenable *locally compact group is important and could be studied,* cf. *[Ped79, Sec. 7.3]. Note that abelian groups and compact groups are amenable.*

3.2 Locally compact abelian groups

We shall now develop the theory of locally compact abelian groups, and refer to [Fol95, Sec. 4] for more details. In particular, one of our aims is to show that the usual Fourier transform is nothing but a Gelfand representation in the context of locally compact abelian groups.

In the section, G will always denote a locally compact abelian group. For them, left and right continuity coincide, convolution is commutative, and the modular function is identically equal to 1. For simplicity, we shall simply denote by $\mathrm{d}x$ a Haar measure on G (which is unique up to a scaling constant), and $L^p(G)$ for $L^p(G, \mathrm{d}x)$ with the norm denoted by $\|\cdot\|_p$.

Let us recall from Corollary 3.1.16 that all unitary irreducible representation of G are one-dimensional. Thus, for each such representation $(\mathcal{H}, U)$ one can take $\mathcal{H} = \mathbb{C}$ and then $U_x = \xi(x)$, where $\xi : G \to \mathbb{T}$ is a continuous homomorphism.

Definition 3.2.1. *For a locally compact abelian group G,* a character *ξ is a continuous homomorphism from G to $\mathbb{T}$. The set of all characters is denoted by $\hat{G}$.*

Note that we shall use both notations $\xi(x)$ or $\langle x, \xi\rangle$. As a consequence of Proposition 3.1.14, this unitary representation induces a non-degenerate representation $(\mathbb{C}, \tau_\xi)$ of $L^1(G)$ by the formula

$$\tau_\xi(f) = \int_G \langle x, \xi\rangle f(x)\mathrm{d}x \tag{3.2.1}$$

for any $f \in L^1(G)$. Since $\mathscr{B}(\mathbb{C})$ is clearly identified with $\mathbb{C}$, such a representation is nothing but a character on the algebra $L^1(G)$, *i.e.* an element of $\Omega\big(L^1(G)\big)$, see Definition 2.3.1. Conversely, any character τ on $L^1(G)$ defines a character on G. Indeed, observe first that any $\tau \in \Omega\big(L^1(G)\big) = L^1(G)^*$ is obtained by integration against some $\xi \in L^\infty(G)$. Then, choose $f \in L^1(G)$ such that $\tau(f) \neq 0$. For any $g \in L^1(G)$ one has

$$\begin{aligned}\tau(f)\int_G \xi(y)g(y)\mathrm{d}y &= \tau(f)\tau(g) = \tau(f * g)\\ &= \int_G\int_G \xi(x)f(xy^{-1})g(y)\mathrm{d}y\,\mathrm{d}x = \int_G \tau(L_y f)g(y)\mathrm{d}y\end{aligned}$$

so that $\xi(y) = \frac{\tau(L_y f)}{\tau(f)}$ locally a.e. We can thus redefine ξ such that $\xi(y) = \frac{\tau(L_y f)}{\tau(f)}$ for every $y \in G$, and then ξ is continuous. As a consequence, one has

$$\xi(xy)\tau(f) = \tau(L_{xy}f) = \tau(L_x L_y f) = \xi(x)\xi(y)\tau(f)$$

which means $\xi(xy) = \xi(x)\xi(y)$. Finally, $\xi(x^n) = \xi(x)^n$ for any $n \in \mathbb{Z}$, and since ξ is bounded it implies that $|\xi(x)| = 1$. As a consequence, ξ is a character on G, as expected.

We have thus proved that:

Theorem 3.2.2. *For any locally compact abelian group, the set of characters $\hat{G}$ can be identified with $\Omega\big(L^1(G)\big)$ through formula* (3.2.1).

$\hat{G}$ is an abelian group under pointwise multiplication, its identity is the constant function 1 on G, and one has

$$\langle x, \xi^{-1}\rangle = \overline{\langle x, \xi\rangle} = \langle x^{-1}, \xi\rangle.$$

By endowing $\hat{G}$ with the weak* topology inherited from $L^\infty(G)$, one infers that $\hat{G}$ is a locally compact abelian group, called *the dual group of* G. Note that this topology coincides with the one borrowed from $\Omega\big(L^1(G)\big)$ through the identification mentioned above.

Examples 3.2.3. *(i) For $G = \mathbb{R}$, $\hat{G} \cong \mathbb{R}$ with the pairing $\langle x, \xi\rangle = e^{i\xi x}$,*

(ii) For $G = \mathbb{T}$, $\hat{G} \cong \mathbb{Z}$ with the pairing $\langle \alpha, n\rangle = \alpha^n$,

(iii) For $G = \mathbb{Z}$, $\hat{G} \cong \mathbb{T}$ with the pairing $\langle n, \alpha\rangle = \alpha^n$.

Let us add some information in the case of compact or discrete groups.

Lemma 3.2.4. *If G is a compact abelian group with a Haar measure normalized such that $\int_G \mathrm{d}x = 1$, then $\hat{G}$ is an orthonormal set in $L^2(G)$.*

Proof. If $\xi \in \hat{G}$ then $|\xi| = 1$ and therefore $\|\xi\|_2 = 1$. If $\xi, \eta \in \hat{G}$ with $\xi \neq \eta$ there exists $x_0 \in G$ such that $\langle x_0, \xi\eta^{-1}\rangle \neq 1$, and then we have

$$\int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x = \langle x_0, \xi\eta^{-1}\rangle \int_G \langle x_0^{-1}x, \xi\eta^{-1}\rangle \mathrm{d}x = \langle x_0, \xi\eta^{-1}\rangle \int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x,$$

which implies that $\int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x = 0$. $\square$

Proposition 3.2.5. *If G is discrete, then $\hat{G}$ is compact. If G is compact, then $\hat{G}$ is discrete.*

Proof. If G is discrete, then $L^1(G)$ has a unit, and therefore $\Omega\big(L^1(G)\big)$ is compact. By Theorem 3.2.2, it follows that $\hat{G}$ is compact.

If G is compact (with Haar measure satisfying $\int_G \mathrm{d}x = 1$), then the constant function 1 belongs to $L^1(G)$. It follows from Lemma 3.2.4 that $\int_G \langle x, \xi\rangle \mathrm{d}x = 1$ if $\xi = 1$ while $\int_G \langle x, \xi\rangle \mathrm{d}x = \langle 1, \xi\rangle_{L^2(G)} = 0$ if $\xi \in \hat{G}$ with $\xi \neq 1$. Since the set $\{f \in L^\infty(G) \mid |\int_G f(x)\mathrm{d}x| > 1/2\}$ is a weak* open set, it follows that $\{1\}$ is an open set in $\hat{G}$, and therefore $\hat{G}$ is discrete. □

Henceforth, it is more convenient (and more common) to use a slightly different identification of $\hat{G}$ with $\Omega\big(L^1(G)\big)$ than the one given in (3.2.1). Namely, we associate with $\xi \in \hat{G}$ the functional

$$f \mapsto \int_G \overline{\langle x, \xi\rangle} f(x)\mathrm{d}x.$$

The Gelfand transform for the abelian Banach algebra $L^1(G)$ becomes then the map $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ defined by

$$[\mathcal{F}f](\xi) \equiv \hat{f}(\xi) = \int_G \overline{\langle x, \xi\rangle} f(x)\mathrm{d}x$$

and is usually called in this context *the Fourier transform.* A rephrasing of Theorem 2.3.5 together with some simple verifications lead to:

Theorem 3.2.6. *The Fourier transform is a norm decreasing $*$-homomorphism from $L^1(G)$ to $C_0(\hat{G})$, or to $C(\hat{G})$ if $\hat{G}$ is compact. It extends to a $*$-isomorphism between $\mathscr{C}^*(G)$ and $C_0(\hat{G})$.*

Extension 3.2.7. *In the setting presented above, many classical results of Fourier analysis on $\mathbb{R}^d$ can be extended to arbitrary locally compact abelian groups. This subject is nicely presented in Section 4 of [Fol95]. A look at Plancherel Theorem, at some Fourier inversions formula or at Pontrjagin duality theorem is certainly valuable.*

3.3 C^*-dynamical systems

In the sequel, we shall go on with the convention of simply writing $\mathrm{d}x$ for a left Haar measure on G, and denote by $L^p(G)$ the spaces $L^p(G, \mathrm{d}x)$.

Definition 3.3.1. A C^*-dynamical system *consists in a triple $(\mathscr{C}, G, \theta)$, where $\mathscr{C}$ is a C^*-algebra, G is a locally compact group, and θ is a continuous homomorphism from G to $\mathsf{Aut}(\mathscr{C})$, with $\mathsf{Aut}(\mathscr{C})$ the group of $*$-automorphisms of $\mathscr{C}$ equipped with the topology of pointwise convergence.*

Note that the topology on $\mathsf{Aut}(\mathscr{C})$ means that for each $A \in \mathscr{C}$, the map

$$G \ni x \mapsto \theta_x(A) \in \mathscr{C}$$

is continuous.

Example 3.3.2. *Let us present an example which will be important later on. We consider the C^*-algebra $\mathscr{C} := BC_u(\mathbb{R}^d)$, the group $G = \mathbb{R}^d$ (with the additive notation) and the action θ of G on $\mathscr{C}$ by translation,* i.e. *$[\theta_x f](y) = f(y - x)$ for any $f \in \mathscr{C}$ and $x, y \in \mathbb{R}^d$. Almost by definition, the algebra $BC_u(\mathbb{R}^d)$ is the largest algebra of functions on $\mathbb{R}^d$ for which this action is continuous, namely $\|\theta_x f - f\|_\infty \to 0$ as $x \to 0$. Then the triple $(\mathscr{C}, G, \theta)$ is a C^*-dynamical system. Note that any C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is stable under translations would also be suitable for such a dynamical system, as for example $C_0(\mathbb{R}^d)$.*

Exercise 3.3.3. *Let G be a locally compact group, Ω be a locally compact space, and assume that the group G acts continuously on Ω,* i.e. *there exists a continuous map*

$$G \times \Omega \ni (x, \xi) \mapsto x \cdot \xi \in \Omega$$

such that $1 \cdot \xi = \xi$ and $x \cdot (y \cdot \xi) = xy \cdot \xi$ for all $x, y \in G$ and $\xi \in \Omega$. Such a system is called a locally compact transformation group*, and Ω is also called* a locally compact G-space*. Then, let us define an automorphism of $C_0(\Omega)$ by $[\theta_x f](\xi) := f(x^{-1} \cdot \xi)$ for any $f \in C_0(\Omega)$, $x \in G$ and $\xi \in \Omega$. Check that the triple $\big(C_0(\Omega), G, \theta\big)$ is a C^*-dynamical system. In fact, it turns out that all C^*-dynamical systems with $\mathscr{C}$ abelian arise from locally compact transformation groups, see [Wil07, Prop. 2.7] for details.*

Definition 3.3.4. A covariant representation *of a C^*-dynamical system $(\mathscr{C}, G, \theta)$ consists in a triple $(\mathcal{H}, \pi, U)$, where $(\mathcal{H}, \pi)$ is a representation of $\mathscr{C}$, $(\mathcal{H}, U)$ is a unitary representation of G, and the following compatibility condition holds*

$$\pi\big(\theta_x(A)\big) = U_x \pi(A) U_x^*$$

for all $A \in \mathscr{C}$ and $x \in G$. This covariant representation is non-degenerate if the representation $(\mathcal{H}, \pi)$ of $\mathscr{C}$ is non-degenerate.

Examples 3.3.5. *Covariant representations of the dynamical systems $(\mathscr{C}, \{1\}, id)$ correspond exactly to representation of $\mathscr{C}$. On the other hand, covariant representations of the dynamical systems $(\mathbb{C}, G, id)$ coincide with unitary representations of G.*

Example 3.3.6 (Regular representation)**.** *Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let $(\mathcal{H}, \pi)$ be a representation of $\mathscr{C}$. Consider the Hilbert space $L^2(G; \mathcal{H}) \cong L^2(G) \otimes \mathcal{H}$, and let us then define $\tilde{\pi} : \mathscr{C} \to \mathscr{B}\big(L^2(G; \mathcal{H})\big)$ and $\tilde{U} : G \to \mathscr{U}\big(L^2(G; \mathcal{H})\big)$ by*

$$[\tilde{\pi}(A)h](x) := \pi\big(\theta_x^{-1}(A)\big)h(x) \qquad \text{and} \qquad [\tilde{U}_x h](y) := h(x^{-1}y),$$

for any $A \in \mathscr{C}$, $h \in L^2(G;\mathcal{H})$ and $x, y \in G$. Let us now check that

$$\begin{aligned}[\tilde{U}_x \tilde{\pi}(A) \tilde{U}_x^* h](y) = [\tilde{\pi}(A)\tilde{U}_x^* h](x^{-1}y) &= \pi\big(\theta^{-1}_{x^{-1}y}(A)\big)\big(\tilde{U}_x^* h(x^{-1}y)\big) \\ &= \pi\big[\theta_y^{-1}\big(\theta_x(A)\big)\big]\big(h(y)\big) = \big[\tilde{\pi}\big(\theta_x(A)\big)h\big](y).\end{aligned}$$

Thus, the triple $\big(L^2(G;\mathcal{H}), \tilde{\pi}, \tilde{U}\big)$ is a covariant representation of the C^*-dynamical system, called its regular representation. *As a consequence, any C^*-dynamical system has at least one covariant representation. It can also be shown that the regular representation is non-degenerate if the representation $(\mathcal{H}, \pi)$ of $\mathscr{C}$ is non-degenerate,* cf. *[Wil07, Lem. 2.17].*

Exercise 3.3.7. *Let G be a locally compact group and its left action on elements of $C_0(G)$,* i.e. *$[L_x f](y) = f(x^{-1}y)$. In this setting, check that $\big(C_0(G), G, L\big)$ is a C^*-dynamical system. Now, let $\mathcal{H} := L^2(G)$ and define $\mathrm{Id} : C_0(G) \to \mathscr{B}(\mathcal{H})$ be the identification map defined by $[\mathrm{Id}(f)h](x) = f(x)h(x)$ for any $f \in C_0(G)$ and $h \in \mathcal{H}$. Finally, let $U_x \in \mathscr{U}(\mathcal{H})$ defined by $[U_x h](y) = h(x^{-1}y)$. Check that $(\mathcal{H}, \mathrm{Id}, U)$ is a covariant representation of $\big(C_0(G), G, L\big)$.*

3.4 Crossed product algebras

This section is mainly based on [Ped79, Sec. 7.6] together with [Wil07, Sec. 2.3]. However, note that quite a lot of explicit computations are explicitly written in [Sko12].

Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let us define a product and an involution on the linear space $C_c(G;\mathscr{C})$ of continuous functions from G to $\mathscr{C}$ with compact support: for any $f, g \in C_c(G;\mathscr{C})$ and $x \in G$ one sets

$$\begin{aligned}[f * g](x) &:= \int_G f(y)\,\theta_y\big(g(y^{-1}x)\big)\,\mathrm{d}y \\ f^*(x) &:= \Delta(x)^{-1}\theta_x\big(f(x^{-1})^*\big).\end{aligned}$$

Some lengthy but straightforward computations show that these definitions endow $C_c(G;\mathscr{C})$ with an associative product and with an involution. In addition, if one sets $\|f\|_1 := \int_G \|f(y)\|\mathrm{d}y$, then $C_c(G;\mathscr{C})$ becomes a norm algebra with a submultiplicative norm, *i.e.* $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. The completion of $C_c(G;\mathscr{C})$ with this norm is denoted by $L^1(G;\mathscr{C})$ which is therefore a B^*-algebra.

Clearly, if $\mathscr{C} = \mathbb{C}$, the above construction leads simply to the algebra $L^1(G)$. Let us also observe that if $f \in L^1(G)$ and $A \in \mathscr{C}$, then the element $f \otimes A$ is an element of $L^1(G;\mathscr{C})$. In addition, the linear span of elements of the form $f \otimes A$ with $f \in C_c(G)$ and $A \in \mathscr{C}$ is dense in $L^1(G;\mathscr{C})$.

Let us now state an important result relating a covariant representation of a C^*-dynamical system to a representation of the corresponding L^1-algebra:

Theorem 3.4.1. *If $(\mathcal{H},\pi,U)$ is a covariant representation of the C^*-dynamical system $(\mathscr{C},G,\theta)$, then there is a norm-decreasing representation $(\mathcal{H},\pi\rtimes U)$ of $L^1(G;\mathscr{C})$ defined by*

$$\pi \rtimes U(f) = \int_G \pi\big(f(y)\big)U_y \mathrm{d}y \tag{3.4.1}$$

for every $f \in C_c(G;\mathscr{C})$. Moreover, $(\mathcal{H},\pi\rtimes U)$ is non-degenerate if $(\mathcal{H},\pi)$ is non-degenerate.

The representation $(\mathcal{H},\pi\rtimes U)$ is called *the integrated representation of* $(\mathcal{H},\pi,U)$. We provide below a sketch of the proof, and refer to [Wil07, Prop. 2.23] for the details.

Proof. Let $f \in C_c(G;\mathscr{C})$ and define $\pi\rtimes U(f) \in \mathscr{B}(\mathcal{H})$ by (3.4.1). Then, one observes that

$$\begin{aligned}
\pi \rtimes U(f^*) &= \int_G \pi\big[\Delta(y)^{-1}\theta_y\big(f(y^{-1})^*\big)\big]U_y \mathrm{d}y \\
&= \int_G \Delta(y)^{-1}U_y\pi\big(f(y^{-1})^*\big)\mathrm{d}y = \int_G U_y^*\pi\big(f(y)^*\big)\mathrm{d}y = \big(\pi\rtimes U(f)\big)^*,
\end{aligned}$$

and (with $g \in C_c(G;\mathscr{C})$)

$$\begin{aligned}
\pi \rtimes U(f*g) &= \int_G \pi\Big[\int_G f(y)\theta_y\big(g(y^{-1}x)\big)\mathrm{d}y\Big]U_x \mathrm{d}x \\
&= \int_G \Big[\int_G \pi\big[f(y)\big]U_y\pi\big[g(y^{-1}x)\big]U_y^*U_x\mathrm{d}y\Big]\mathrm{d}x \\
&= \int_G \Big[\int_G \pi\big[f(y)\big]U_y\pi\big[g(x)\big]U_x\mathrm{d}y\Big]\mathrm{d}x \\
&= \pi\rtimes U(f)\ \pi\rtimes U(g).
\end{aligned}$$

In addition, one also has $\|\pi\rtimes U(f)\| \leq \int_G \big\|\pi\big(f(y)\big)U_y\big\|\mathrm{d}y = \|f\|_1$. These relations show that $\big(\mathcal{H},\pi\rtimes U\big)$ extends to a norm-decreasing representation of $L^1(G;\mathscr{C})$.

For the non-degeneracy, we refer to the proof of [Wil07, Prop. 2.23]. $\square$

Definition 3.4.2. *For any C^*-dynamical system $(\mathscr{C},G,\theta)$ and any $f \in C_c(G;\mathscr{C})$ let us set*

$$\|f\| := \sup\{\|\pi\rtimes U(f)\|_{\mathscr{B}(\mathcal{H})} \mid (\mathcal{H},\pi,U) \textit{ is a covariant representation of } (\mathscr{C},G,\theta)\}. \tag{3.4.2}$$

The norm $\|\cdot\|$ on $C_c(G;\mathscr{C})$ is called the universal norm*, and is dominated by the $\|\cdot\|_1$-norm. The completion of $C_c(G;\mathscr{C})$ with respect to the norm $\|\cdot\|$ is called* the crossed product C^*-algebra of $\mathscr{C}$ by G *and is denoted by $\mathscr{C}\rtimes_\theta G$.*

Example 3.4.3. *If G is a locally compact group and if θ corresponds to the left action on $C_0(G)$,* i.e. *$[\theta_x f](y) = f(x^{-1}y)$ for all $f \in C_0(G)$, then $C_0(G)\rtimes_\theta G$ is $*$-isomorphic to the compact operators on $L^2(G)$.*

Remark 3.4.4. *If the C^*-algebra $\mathscr{C}$ is abelian, with $\mathscr{C} \cong C_0(\Omega)$, the corresponding crossed product algebra $\mathscr{C} \rtimes_\theta G$ is also called* transformation group C^*-algebra. *Moreover, it is possible to describe the $*$-algebraic structure on $C_c\big(G; C_0(\Omega)\big)$ in terms of functions on $G \times \Omega$. Indeed, observe first that by obvious identifications one has*

$$C_c(G \times \Omega) \subset C_c\big(G; C_c(\Omega)\big) \subset C_c\big(G; C_0(\Omega)\big).$$

Then, if one denotes the action of G on Ω by $\cdot$ (note that such an action always exists, see Proposition 2.7 of [Wil07]) one ends up with the following formula:

$$[f * g](x, \xi) = \int_G f(y, \xi)\, g(y^{-1}x, y^{-1} \cdot \xi) \mathrm{d}y$$
$$f^*(x, \xi) = \Delta(x)^{-1} \overline{f(x^{-1}, x^{-1} \cdot \xi)}$$

for $f, g \in C_c(X \times \Omega)$ and $(x, \xi) \in G \times \Omega$.

Except in some very special cases, the crossed product algebra $\mathscr{C} \rtimes_\theta G$ contains neither a copy of the algebra $\mathscr{C}$ nor a copy of $L^1(G)$. However, its multiplier algebra $\mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$ does, as we shall observe now. Indeed, for any $A \in \mathscr{M}(\mathscr{C})$, $\mu \in M(G)$ and $f \in C_c(G; \mathscr{C})$ let us define

$$[L_{(A,\mu)} f](x) := A \int_G \theta_y\big(f(y^{-1}x)\big)\, \mathrm{d}\mu(y)$$
$$[R_{(A,\mu)} f](x) := \int_G f(xy^{-1}) \theta_{xy^{-1}}(A) \Delta(y)^{-1}\, \mathrm{d}\mu(y).$$

One can check that $L_{(A,\mu)}$ and $R_{(A,\mu)}$ are bounded by $\|A\|\,\|\mu\|$, and thus extend by continuity to linear operators on $L^1(G; \mathscr{C})$. In addition, some straightforward computations (see [Ped79, Lem. 7.6.3]) show that

$$L_{(A,\mu)}(f * g) = (L_{(A,\mu)} f) * g, \qquad R_{(A,\mu)}(f * g) = f * (R_{(A,\mu)} g)$$

and that $(R_{(A,\mu)} f) * g = f * (L_{(A,\mu)} g)$. Thus, the pair $(L_{(A,\mu)}, R_{(A,\mu)})$ defines a double centralizer on the B^*-algebra $L^1(G; \mathscr{C})$, see Section 2.4. With these notions at hand, one can deduce that:

Theorem 3.4.5. *For any C^*-dynamical system $(\mathscr{C}, G, \theta)$, there exist a non-degenerate faithful $*$-homomorphism*

$$i_{\mathscr{C}} : \mathscr{C} \to \mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$$

and an injective homomorphism

$$i_G : G \to \mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$$

defined by the formulas $i_{\mathscr{C}}(A) := (L_{(A,\delta_1)}, R_{(A,\delta_1)})$ and $i_G(x) := (L_{(\mathbf{1},\delta_x)}, R_{(\mathbf{1},\delta_x)})$.

Proof. See [Ped79, Sec. 7.6] and Proposition 2.34 of [Wil07] for the details. $\square$

By using the alternative representation of the multiplier algebra, as introduced at the end of Chapter 2, one also infers the following corollary:

Corollary 3.4.6. *Let $(\mathcal{H}, \pi, U)$ be a non-degenerate covariant representation of the C^*-dynamical system $(\mathscr{C}, G, \theta)$ such that the representation π of $\mathscr{C}$ is faithful. Then, the maps $\mathscr{C} \ni A \mapsto \pi(A) \in \mathscr{B}(\mathcal{H})$ and $G \ni x \mapsto U_x \in \mathscr{U}(\mathcal{H})$ are injective homomorphisms into $\mathscr{M}\big(\pi \rtimes U(\mathscr{C} \rtimes_\theta G)\big) \subset \mathscr{B}(\mathcal{H})$.*

Proof. The mentioned formula are obtained from the previous theorem by observing that for any $f \in C_c(G, \mathscr{C})$ one has

$$\pi \rtimes U(L_{(A,\delta_1)} f) = \int_G \pi\big([L_{(A,\delta_1)} f](x)\big) U_x \, \mathrm{d}x = \int_G \pi\big(A f(x)\big) U_x \, \mathrm{d}x = \pi(A) \pi \rtimes U(f),$$

and

$$\begin{aligned} \pi \rtimes U(L_{(\mathbf{1},\delta_x)} f) &= \int_G \pi\big([L_{(\mathbf{1},\delta_x)} f](y)\big) U_y \, \mathrm{d}y = \int_G \pi\big(\theta_x f(x^{-1}y)\big) U_y \, \mathrm{d}y \\ &= \int_G U_x \pi\big(f(x^{-1}y)\big) U_x^* U_y \, \mathrm{d}y = \int_G U_x \pi\big(f(y)\big) U_y \, \mathrm{d}y = U_x \pi \rtimes U(f). \end{aligned}$$

□

Remark 3.4.7. *In the context of the previous corollary and by starting again with the double centralizer $(L_{(A,\mu)}, R_{(A,\mu)})$ as above, with $A = \mathbf{1}$ and μ an element of $L^1(G)$, one can also infer that there exists a $*$-homomorphism $i_G : L^1(G) \to \mathscr{M}\big(\pi \rtimes U(\mathscr{C} \rtimes_\theta G)\big) \subset \mathscr{B}(\mathcal{H})$ such that one has $i_G(f) = \int_G f(x) i_G(x) \, \mathrm{d}x$ for any $f \in L^1(G)$. In fact, this $*$-homomorphism continuously extends to a $*$-homomorphism from $\mathscr{C}^*(G)$ to the multiplier algebra $\mathscr{M}\big(\pi \rtimes U(\mathscr{C} \rtimes_\theta G)\big)$.*

By using the multiplier algebra and the two maps introduced above, it is rather straightforward to improve Theorem 3.4.1:

Theorem 3.4.8. *For any C^*-dynamical system $(\mathscr{C}, G, \theta)$, the map sending a covariant representation $(\mathcal{H}, \pi, U)$ to the integrated form $(\mathcal{H}, \pi \rtimes U)$ is a bijective correspondence between non-degenerate covariant representations of $(\mathscr{C}, G, \theta)$ and non-degenerate representations of $\mathscr{C} \rtimes_\theta G$.*

We stress that this theorem asserts in particular that any representation of the C^*-algebra $\mathscr{C} \rtimes_\theta G$ corresponds to the integrated form of a covariant representation of the underlying dynamical system. Let us now end this section with a technical result which will be important later on. Its proof is not complicated but is based on some preliminary results which are not trivial, see Lemma 2.45 and Corollary 2.48 of [Wil07]. Note that in this section, most of the difficulties do not come from the algebraic computations but from some topological considerations.

Lemma 3.4.9. *Let $(\mathscr{C}^1, G, \theta^1)$ and $(\mathscr{C}^2, G, \theta^2)$ be C^*-dynamical systems, and let $\varphi : \mathscr{C}^1 \to \mathscr{C}^2$ be an equivariant $*$-homomorphism*[3]*. Then there is a $*$-homomorphism*

$$\varphi \rtimes \mathrm{id} : \mathscr{C}^1 \rtimes_{\theta^1} G \to \mathscr{C}^2 \rtimes_{\theta^2} G$$

mapping $C_c(G; \mathscr{C}^1)$ into $C_c(G; \mathscr{C}^2)$ and such that $[\varphi \rtimes \mathrm{id}(f)](x) = \varphi\big(f(x)\big)$ for any $f \in C_c(G; \mathscr{C}^1)$ and $x \in G$.

Extension 3.4.10. *Consider the special case $\mathscr{C} = C(\mathbb{T})$, $G = \mathbb{Z}$ and $[\theta_n f](z) = f(e^{i2\pi n\vartheta} z)$ for any $f \in \mathscr{C}$, $z \in \mathbb{T}$ and some fixed $\vartheta \in [0,1]$. Depending if ϑ is rational or irrational, the corresponding algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is called* the rational or irrational rotation algebra*. Its study has been a hot topic in the early 80's, and continues to be of interest. Some preliminary information can be grasp for example in [Wil07, Prop. 2.56] and in many other references.*

3.5 Invariant ideals and crossed product

Let us consider a C^*-dynamical system $(\mathscr{C}, G, \theta)$, and let $\mathscr{J}$ be a closed and self-adjoint ideal in $\mathscr{C}$ which is θ-invariant ($\Leftrightarrow \theta_x(A) \in \mathscr{J}$ for any $A \in \mathscr{J}$ and $x \in G$). Then, each θ_x restricts to a $*$-automorphism of $\mathscr{J}$, and this defines a C^*-dynamical system $(\mathscr{J}, G, \theta)$ as well as a quotient C^*-dynamical system $(\mathscr{C}/\mathscr{J}, G, \theta)$, where

$$\theta_x(A + \mathscr{J}) = \theta_x(A) + \mathscr{J}.$$

Note that we have kept the same notation for the $*$-automorphism θ_x acting on $\mathscr{J}$ and for its action on the quotient algebra $\mathscr{C}/\mathscr{J}$. Since the inclusion map $\iota : \mathscr{J} \to \mathscr{C}$ and the quotient map $q : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ are equivariant $*$-homomorphisms, they define by Lemma 3.4.9 $*$-homomorphisms $\iota \rtimes \mathrm{id} : \mathscr{J} \rtimes_\theta G \to \mathscr{C} \rtimes_\theta G$ and $q \rtimes \mathrm{id} : \mathscr{C} \rtimes_\theta G \to (\mathscr{C}/\mathscr{J}) \rtimes_\theta G$.

Clearly, $C_c(G; \mathscr{J})$ is a self-adjoint ideal in $C_c(G; \mathscr{C})$, and therefore its closure is an ideal in $\mathscr{C} \rtimes_\theta G$, which corresponds to the image of $\mathscr{J} \rtimes_\theta G$ through the $*$-homomorphism $\iota \rtimes \mathrm{id}$. In addition, it can be shown that $\iota \rtimes \mathrm{id}$ is isometric on $C_c(G; \mathscr{J})$, which implies that $\iota \rtimes \mathrm{id}$ is in fact a $*$-isomorphism onto the closure of $C_c(G; \mathscr{J})$ in $\mathscr{C} \rtimes_\theta G$, see [Wil07, Lem. 3.17] for the proof of the isometry.

Let us now state an important result about the functoriality of the crossed product:

Lemma 3.5.1. *Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let $\mathscr{J}$ be a self-adjoint closed ideal in $\mathscr{C}$ which is θ-invariant. Then we have the following short sequence of C^*-algebras:*

$$0 \longrightarrow \mathscr{J} \rtimes_\theta G \xrightarrow{\iota \rtimes \mathrm{id}} \mathscr{C} \rtimes_\theta G \xrightarrow{q \rtimes \mathrm{id}} (\mathscr{C}/\mathscr{J}) \rtimes_\theta G \longrightarrow 0.$$

[3] In the present context, the $*$-homomorphism φ is equivariant if $\varphi\big(\theta^1_x(A)\big) = \theta^2_x\big(\varphi(A)\big)$ for all $A \in \mathscr{C}^1$ and $x \in G$.

The fact that $\iota \rtimes \mathrm{id}$ is a $*$-isomorphism has already been mentioned in the paragraph preceding the statement. Thus it only remains to show that

$$\mathsf{Ker}\left(q \rtimes \mathrm{id}\right) = \iota \rtimes \mathrm{id}\big(\mathscr{J} \rtimes_\theta G\big)$$

which can be achieved with the use of an approximate unit, see [Wil07, Prop. 3.19] for the details.

Let us close this section by considering the previous result in the context of transformation group C^*-algebras, see Remark 3.4.4. More precisely, let us consider the C^*-dynamical system $(C_0(\Omega), G, \theta)$ with $[\theta_x f](\xi) = f(x^{-1} \cdot \xi)$ for any $f \in C_0(\Omega)$, $x \in G$ and $\xi \in \Omega$. In this framework, the θ-invariant ideals of $C_0(\Omega)$ corresponds to subalgebras $C_0(\Omega')$ with Ω' a G-invariant open subset of Ω. Then, let us set $F := \Omega \setminus \Omega'$, which is a G-invariant closed subset of Ω, and let us identify $C_0(F)$ with the quotient $C_0(\Omega)/C_0(\Omega')$ (notice that the $*$-homomorphism $q : C_0(\Omega) \to C_0(F)$ is equivariant). A special case of the previous lemma reads then:

Corollary 3.5.2. *Let us consider the C^*-dynamical system $(C_0(\Omega), G, \theta)$, and let Ω' be an open G-invariant subset of Ω. Then we have the following short exact sequence of C^*-algebras*

$$0 \longrightarrow C_0(\Omega') \rtimes_\theta G \xrightarrow{\iota \rtimes \mathrm{id}} C_0(\Omega) \rtimes_\theta G \xrightarrow{q \rtimes \mathrm{id}} C_0(\Omega \setminus \Omega') \rtimes_\theta G \longrightarrow 0. \qquad (3.5.1)$$