Characters, Character sheaves and Beyond

Toshiaki Shoji

Nagoya University

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Kostka polynomials $K_{\lambda,\mu}(t)$

$\lambda = (\lambda_1, \dots, \lambda_k)$: **partition of n**

$\lambda_i \in \mathbb{Z}_{\geq 0}, \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_i \lambda_i = n$

$\mathcal{P}_n = \{\text{partitions of } n\}$

$s_\lambda(x) = s_\lambda(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$: **Schur function**

$$s_\lambda(x) = \det(x_i^{\lambda_j + k - j})_{1 \leq i,j \leq k} / \det(x_i^{k-j})_{1 \leq i,j, \leq k}$$

$P_\lambda(x;t) = P_\lambda(x_1, \dots, x_k; t) \in \mathbb{Z}[x_1, \dots, x_k; t]$: **Hall-Littlewood function**

$$P_\lambda(x_1, \dots, x_k; t) = \sum_{w \in S_k / S_k^\lambda} w \left(x_1^{\lambda_1} \cdots x_k^{\lambda_k} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

Take $k \gg 0$

$\{s_\lambda(x) \mid \lambda \in \mathcal{P}_n\}, \{P_\lambda(x;t) \mid \lambda \in \mathcal{P}_n\}$ are basis of the space of homog. symmetric polynomilas of degree n in $\mathbb{Z}[x_1, \dots, x_k : t]$

For $\lambda, \mu \in \mathcal{P}_n$, $K_{\lambda,\mu}(t)$: **Kostka polynomial** defined by

$$s_\lambda(x) = \sum_{\mu \in \mathcal{P}_n} K_{\lambda,\mu}(t) P_\mu(x;t)$$

$K_{\lambda,\mu}(t) \in \mathbb{Z}[t]$

$(K_{\lambda,\mu}(t))_{\lambda,\mu \in \mathcal{P}_n}$: transition matrix of two basis $\{s_\lambda(x)\}$, $\{P_\mu(x;t)\}$

Geometric realization of Kostka polynomials

In 1981, Lusztig gave a geometric realization of Kostka polynomials in connection with the closure of nilpotent orbits.

$V = \mathbb{C}^n, \quad G = GL(V)$
$\mathcal{N} = \{x \in \mathrm{End}(V) \mid x : \text{ nilpotent } \}$: nilpotent cone

$$\mathcal{P}_n \simeq \mathcal{N}/G$$

$$\lambda \leftrightarrow G\text{-orbit } \mathcal{O}_\lambda \ni x : \text{ Jordan type } \lambda$$

- **Closure relations :**

$$\overline{\mathcal{O}}_\lambda = \coprod_{\mu \le \lambda} \mathcal{O}_\mu \quad (\overline{\mathcal{O}}_\lambda : \text{ Zariski closure of } \mathcal{O}_\lambda)$$

dominance order on $\mathcal{P}_n$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \mu = (\mu_1, \mu_2, \dots, \mu_k)$,

$\mu \le \lambda \Leftrightarrow \sum_{i=1}^j \mu_i \le \sum_{i=1}^j \lambda_i \quad$ for each j.

Notation: $n(\lambda) = \sum_{i \geq 1}(i-1)\lambda_i$

Define $\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$: **modified Kostka polynomial**

$K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \mathbb{C})$: Intersection cohomology complex

$$K : \cdots \longrightarrow K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \xrightarrow{d_{i+1}} \cdots$$

$K = (K_i)$: bounded complex of $\mathbb{C}$-sheaves on $\overline{\mathcal{O}}_\lambda$

$\mathcal{H}^i K = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$: i-th cohomology sheaf

$\mathcal{H}^i_x K$: the stalk at $x \in \overline{\mathcal{O}}_\lambda$ of $\mathcal{H}^i K$ (finite dim. vecotr space over $\mathbb{C}$)

Known fact : $\mathcal{H}^i K = 0$ for odd i.

Theorem (Lusztig)

For $x \in \mathcal{O}_\mu$,

$$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathcal{H}^{2i}_x K) t^i$$

In particular, $K_{\lambda,\mu}[t] \in \mathbb{Z}_{\geq 0}[t]$. (theorem of Lascoux-Schützenberger)

Representation theory of $GL_n(\mathbb{F}_q)$

$\mathbb{F}_q$: finite field of q elements with $\operatorname{ch}\mathbb{F}_q = p$
$\overline{\mathbb{F}}_q$: algebraic closure of $\mathbb{F}_q$

$$G = GL_n(\overline{\mathbb{F}}_q) \supset B = \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \supset U = \left\{ \begin{pmatrix} 1 & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

B : Borel subgroup, U : maximal unipotent subgroup

$F : G \to G, (g_{ij}) \mapsto (g_{ij}^q)$: Frobenius map
$G^F = \{g \in G \mid F(g) = g\} = G(\mathbb{F}_q)$: finite subgroup

$\operatorname{Ind}_{B^F}^{G^F} 1$: the character of G^F obtained by inducing up 1_{B^F}

$$\operatorname{Ind}_{B^F}^{G^F} 1 = \sum_{\lambda \in \mathcal{P}_n} (\deg \chi^\lambda)\rho^\lambda,$$

ρ^λ: irreducible character of G^F corresp. to $\chi^\lambda \in S_n^\wedge \simeq \mathcal{P}_n$.

$G_{\rm uni} = \{g \in G \mid u\text{: unipotent}\} \subset G, \quad G_{\rm uni} \simeq \mathcal{N}, u \leftrightarrow u - 1$

- $G_{\rm uni}/G \simeq \mathcal{P}_n, \quad \mathcal{O}_\lambda \leftrightarrow \lambda$

$\mathcal{O}_\lambda$: F-stable $\Longrightarrow \mathcal{O}_\lambda^F$: single G^F-orbit, $u_\lambda \in \mathcal{O}_\lambda^F$

Theorem (Green)

$$\rho^\lambda(u_\mu) = \widetilde{K}_{\lambda,\mu}(q)$$

Remark : Lusztig's result $\Longrightarrow$ the character values of ρ^λ at **unipotent elements** are described in terms of intersection cohomology complex.

Theory of character sheaves $\Longrightarrow$ describes **all the character values** of ρ^λ in terms of certain simple perverse sheaves.

Character sheaves on GL_n

X: alg. variety over $\mathbb{F}_q$ with Frobenius map $F : X \to X$

K : perverse sheaf on X, K : F-stable $\Leftrightarrow F^*K \simeq K$.

For F-stable perverse sheaf K, fix $\varphi : F^*K \to K$ isomorphism

Define $\chi_{K,\varphi} : X^F \to \bar{\mathbb{Q}}_l$ by

$$\chi_{K,\varphi}(x) = \sum_i (-1)^i \operatorname{Tr}(\varphi, \mathcal{H}^i_x K) \qquad (x \in X^F)$$

$\chi_{K,\varphi}$: **Characteristic function** of K with respect to φ.

- If K : G-equiv. perverse sheaf $\Rightarrow \chi_{K,\varphi}$: G^F-invariant function on X^F.

Lusztig : All the irreducible characters of $G^F = GL_n(\mathbb{F}_q)$ are obtained as characteristic functions of certain G-equivariant F-stable simple perverse sheaves (i.e., **character sheaves**) of G

Representation theory of finite reductive groups

Green (1955) : classified all irreducible representations of G^F on $\mathbb{C}$ (or $\bar{\mathbb{Q}}_l$), and determined irreducible characters

basic tool : $R_T^G(\theta)$ Green's basic function (Deligne-Lusztig's virtual character)

$$Q_T^G := R_T^G(\theta)|_{G^F_{\text{uni}}} : G^F_{\text{uni}} \to \bar{\mathbb{Q}}_l \quad \textbf{Green function}$$

- $\pm R_T^G(\theta)$: irreducible character for generic pair (T, θ)
- Any irreducible character is a linear combination of $R_T^G(\theta)$
- Computation of $R_T^G(\theta) \Longleftarrow$ computaiton of Green functions

Green functions are described by Kostka polynomials !!

Deligne-Lusztig (1976) : Deligne-Lusztig's virtual rep. $R_T^G(\theta)$ for connected reductive groups by using ℓ-adic cohomology theory

Lusztig (1980's) : Classification of irreducible representations for connected reductive groups

Lusztig (1985) : Theory of character sheaves (geometric theory of characters of reductive groups)

Lusztig's conjecture : Uniform algorithm of computing irreducible characters (in principle)

S (1995) : Solved Lusztig's conjecture in the case where the center of G is connected

Bonnafe, S, Waldspurger (2004 $\sim$) : Lusztig's conjectue for disconnected center case for SL_n, Sp_{2n}, SO_{2n} (open in general)

Enhanced nilpotent cone

$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}), \quad \sum_{i=1}^{r} |\lambda^{(i)}| = n$: **r-partition of n**

$\mathcal{P}_{n,r}$: the set of r-partitions of n

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,r}$, one can define $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbb{Q}(t)$: **Kostka functions associated to complex reflection groups**.

Achar-Henderson (2008) : geometric realization of Kostka functions for $r = 2$

$V = \mathbb{C}^n, \quad \mathcal{N}$: nilpotent cone
$\mathcal{N} \times V$: **enhanced nilpotent cone**, diagonal action of $G = GL(V)$

Achar-Henderson, Travkin :

$$(\mathcal{N} \times V)/G \simeq \mathcal{P}_{n,2}, \quad \mathcal{O}_{\boldsymbol{\lambda}} \leftrightarrow \boldsymbol{\lambda}$$

- **Closure relations** $\quad \overline{\mathcal{O}}_{\boldsymbol{\lambda}} = \coprod_{\boldsymbol{\mu} \le \boldsymbol{\lambda}} \mathcal{O}_{\boldsymbol{\mu}}$

$K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \mathbb{C})$: Intersection cohomology complex

Theorem (Achar-Henderson)

$\mathcal{H}^i K = 0$ for odd i. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2}$, and $(x, v) \in \mathcal{O}_{\boldsymbol{\mu}} \subseteq \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$,

$$t^{a(\boldsymbol{\lambda})} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathcal{H}^{2i}_{(x,v)} K) t^{2i} = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t),$$

where $a(\boldsymbol{\lambda}) = 2n(\lambda^{(1)}) + 2n(\lambda^{(2)}) + |\lambda^{(2)}|$ for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$.

$\mathcal{N} \times V \rightsquigarrow G_{\mathrm{uni}} \times V \hookrightarrow G \times V$ (over $\overline{\mathbb{F}}_q$) : diagonal action of $G = GL(V)$

Finkelberg-Ginzburg-Travkin (2008) : Theory of character sheaves on $G \times V$ (certain G-equiv. simple perverse sheaves)

$\Longrightarrow$ **"character table"** of $(G \times V)^F$

S (2010) : Generalization to $(G \times V^{r-1})^F$, in connection with Kostka functions assoc. to $\mathcal{P}_{n,r}$ (in progress)

Finite symmetric space $GL_{2n}(\mathbb{F}_q)/Sp_{2n}(\mathbb{F}_q)$

$G = GL(V) \simeq GL_{2n}(\overline{\mathbb{F}}_q), \quad V = (\overline{\mathbb{F}}_q)^{2n}, \quad \mathrm{ch}\,\mathbb{F}_q \neq 2$

$\theta : G \to G$, $\theta(g) = J^{-1}({}^t g^{-1})J$: involution, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$K := \{g \in G \mid \theta(g) = g\} \simeq Sp_{2n}(\overline{\mathbb{F}}_q) \quad G/K$: symmetric space over $\overline{\mathbb{F}}_q$

$G^F \simeq GL_{2n}(\mathbb{F}_q) \supset Sp_{2n}(\mathbb{F}_q) \simeq K^F$

G^F acts on $G^F/K^F \rightsquigarrow 1_{K^F}^{G^F}$: induced representation

$H(G^F, K^F) := \mathrm{End}_{G^F}(1_{K^F}^{G^F})$: **Hecke algebra** asoc. to (G^F, K^F)

- $H(G^F, K^F)$: commutative algebra
- $H(G^F, K^F)^\wedge$: natural labeling by $(GL_n^F)^\wedge$
- $K^F\backslash G^F/K^F$: natural labeling by $\{$ conj. classes of$GL_n^F\}$

Theorem (Bannai-Kawanka-Song, 1990)

The character table of $H(G^F, K^F)$ can be obtained from the character table of GL_n^F by replacing $q \mapsto q^2$.
More precisely, there exist **basic functions, Green functions** assoc. to $H(G^F, K^F)$, which have the same role as those for $GL_n(\mathbb{F}_q)$.

Geometric setting for G/K

$$\begin{aligned} G^{\iota\theta} &= \{g \in G \mid \theta(g) = g^{-1}\} \\ &= \{g\theta(g)^{-1} \mid g \in G\}, \end{aligned}$$

where $\iota : G \to G, g \mapsto g^{-1}$.

The map $G \to G,\ g \mapsto g\theta(g)^{-1}$ gives isom. $G/K \xrightarrow{\sim} G^{\iota\theta}$.

K acts by left mult $\curvearrowright G/K \simeq G^{\iota\theta} \curvearrowleft K$ acts by conjugation.

$$K\backslash G/K \simeq \{K\text{-conjugates of } G^{\iota\theta}\}$$

- Irred. character of $H(G^F, K^F) \Longleftrightarrow K^F$-inv. function on $(G^{\iota\theta})^F$
- character sheaves $\Longleftrightarrow$ K-equiv. simple perverse sheaves on $G^{\iota\theta}$

Henderson : Geometric reconstruction of BKS (not complete)

Lie algebra analogue

$\mathfrak{g} = \mathfrak{gl}_{2n}, \quad \theta : \mathfrak{g} \to \mathfrak{g}$: involution, $\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{g}^{-\theta}$,

$\mathfrak{g}^{\pm\theta} = \{x \in \mathfrak{g} \mid \theta(x) = \pm x\}$, $\quad K$-stable subspace of $\mathfrak{g}$

$\mathfrak{g}^{-\theta}_{\mathrm{nil}} = \mathfrak{g}^{-\theta} \cap \mathcal{N}_{\mathfrak{g}}$: analogue of nilpotent cone $\mathcal{N}$, K-stable subset of $\mathfrak{g}^{-\theta}$

$$\mathfrak{g}^{-\theta}_{\mathrm{nil}}/K \simeq \mathcal{P}_n, \quad \mathcal{O}_\lambda \leftrightarrow \lambda$$

Theorem (Henderson + BKS, 2008)

Let $K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbb{Q}}_l)$, $x \in \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$. Then $\mathcal{H}^i K = 0$ unless $i \equiv 0 \pmod 4$, and

$$t^{2n(\lambda)} \sum_{i \ge 0} (\dim \mathcal{H}^{4i}_x K) t^{2i} = \widetilde{K}_{\lambda,\mu}(t^2)$$

Exotic symmetric space $GL_{2n}/Sp_{2n} \times V$

(Joint work with K. Sorlin)

Bannai-Kawanaka-Song

Green, Lusztig

Henderson

$$\begin{tikzcd} GL_n \arrow[r] \arrow[d] & GL_{2n}/Sp_{2n} \arrow[d] \\ GL_n \times V_n \arrow[r] & GL_{2n}/Sp_{2n} \times V_{2n} \end{tikzcd}$$

???

Achar-Henderson

Finkelberg-Ginzburg-Travkin

Kato's exotic nilcone

$G = GL(V) \simeq GL_{2n}(\overline{\mathbb{F}}_q), \quad \dim V = 2n, \quad K = G^{\theta}.$
$G^{\iota\theta} \times V$: K acts diagonally

Problem

- Find a good class of K-equivariant simple perverse sheaves on $G^{\iota\theta} \times V$, i.e., **"character sheaves"** on $G^{\iota\theta} \times V$
- Find a good basis of K^F-equivariant functions on $(G^{\iota\theta} \times V)^F$, i.e., **"irreducible characters"** of $(G^{\iota\theta} \times V)^F$, and compute their values, i.e., computaion of the **"character table"**

Remark : $\mathcal{X}_{\text{uni}} := G^{\iota\theta}_{\text{uni}} \times V \simeq \mathfrak{g}^{-\theta}_{\text{nil}} \times V$: **Kato's exotic nilcone**

Kato $(\mathfrak{g}^{-\theta}_{\text{nil}} \times V)/K \simeq \mathcal{P}_{n,2}, \quad \mathcal{O}_{\boldsymbol{\mu}} \leftrightarrow \boldsymbol{\mu} \in \mathcal{P}_{n,2}$

Natural bijection with GL_n-orbits of enhanced nilcone, compatible with closure relations (**Achar-Henderson**)

Springer correspondence

$B = TU$: θ-stable Borel subgroup, maximal torus, unipotent radical

$M_0 \subset M_1 \subset \cdots \subset M_n$: isotorpic flag stable by B

$W_n = N_K(T^\theta)/T^\theta$: Weyl group of type C_n

$$\widetilde{\mathcal{X}}_{\text{uni}} = \{(x, v, gB^\theta) \in G^{\iota\theta}_{\text{uni}} \times V \times K/B^\theta \mid (g^{-1}xg, g^{-1}v) \in U^{\iota\theta} \times M_n\}$$
$$\pi_1 : \widetilde{\mathcal{X}}_{\text{uni}} \to \mathcal{X}_{\text{uni}}, \quad (x, v, gB^\theta) \mapsto (x, v)$$

Theorem 1 (Springer correspondence)

$(\pi_1)_*\bar{\mathbb{Q}}_l$ is a semisimple perverse sheaf on $\mathcal{X}_{\text{uni}}$ with W_n-action, is decomposed as

$$(\pi_1)_*\bar{\mathbb{Q}}_l[\dim \mathcal{X}_{\text{uni}}] \simeq \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} V_{\boldsymbol{\mu}} \otimes \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\mu}^\bullet}, \bar{\mathbb{Q}}_l)[\dim \mathcal{O}_{\boldsymbol{\mu}^\bullet}],$$

$V_{\boldsymbol{\mu}}$: standard irred. W_n-module, $\mathcal{O}_{\boldsymbol{\mu}^\bullet} \mapsto V_{\boldsymbol{\mu}}$ gives bijection $\mathcal{X}_{\text{uni}}/K \simeq W_n^\wedge$

Remark : Theorem 1 was first proved by Kato for the exotic nilcone by using Ginzburg theroy on affine Hecke algebras. We give an alternate proof based on the theory of character sheaves

$$T^{\iota\theta}_{\mathrm{reg}} = \{t = \mathrm{Diag}(t_1, \dots, t_n, t_1, \dots, t_n) \mid t_i \neq t_j\}$$

$$\widetilde{G}^{\iota\theta}_{\mathrm{reg}} = \{(x, gB^\theta) \in G^{\iota\theta} \times K/B^\theta \mid g^{-1}xg \in T^{\iota\theta}_{\mathrm{reg}}\}$$

$$\psi_0 : \widetilde{G}^{\iota\theta}_{\mathrm{reg}} \to G^{\iota\theta}_{\mathrm{reg}} = \bigcup_{g\in K} g(T^{\iota\theta}_{\mathrm{reg}})g^{-1}, \quad (x, gB^\theta) \mapsto x$$

$$\widetilde{G}^{\iota\theta}_{\mathrm{reg}} \simeq K \times^{(Z_K(T^{\iota\theta})\cap B^\theta)} T^{\iota\theta}_{\mathrm{reg}} \xrightarrow{\xi} K \times^{Z_K(T^{\iota\theta})} T^{\iota\theta}_{\mathrm{reg}} \xrightarrow{\eta} G^{\iota\theta}_{\mathrm{reg}}$$

- η is a finite Galois covering with group $S_n \simeq N_K(T^{\iota\theta})/Z_K(T^{\iota\theta})$
- ξ is a $\mathbb{P}^n_1$-bundle, with $\mathbb{P}^n_1 \simeq (SL_2/B_2)^n$

$$(\psi_0)_*\bar{\mathbb{Q}}_l \simeq H^\bullet(\mathbb{P}_1^n, \bar{\mathbb{Q}}_l) \otimes \eta_*\bar{\mathbb{Q}}_l$$

- η : fintie Galois covering $\Longrightarrow \eta_*\bar{\mathbb{Q}}_l$ has a natural action of S_n
- $(\mathbb{Z}/2\mathbb{Z})^n$: Weyl group of $(SL_2)^n$ acts on $H^\bullet(\mathbb{P}_1^n, \bar{\mathbb{Q}}_l)$
- $(\psi_0)_*\bar{\mathbb{Q}}_l$ has a natural action of $W_n = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$

For $0 \le m \le n$,

$$\widetilde{\mathcal{Y}}_m = \{(x, v, gB^\theta) \in G^{\iota\theta}_{\mathrm{reg}} \times V \times K/B^\theta \mid (g^{-1}xg, g^{-1}v) \in B^{\iota\theta}_{\mathrm{reg}} \times M_m\}$$

$$\mathcal{Y}_m = \bigcup_{g \in K} g(B^{\iota\theta}_{\mathrm{reg}} \times M_m)$$

$$\psi : \widetilde{\mathcal{Y}}_n \to \mathcal{Y}_n = G^{\iota\theta}_{\mathrm{reg}} \times V, \quad (x, v, gB^\theta) \mapsto (x, v)$$

Proposition 1

$$\psi_*\bar{\mathbb{Q}}_l[\dim \mathcal{Y}_n] \simeq \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} V_{\boldsymbol{\mu}} \otimes \mathrm{IC}(\mathcal{Y}_{m(\boldsymbol{\mu})}, \mathcal{L}_{\boldsymbol{\mu}})[\dim \mathcal{Y}_{m(\boldsymbol{\mu})}],$$

$m(\boldsymbol{\mu}) = |\mu^{(1)}|$ for $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$, $\mathcal{L}_{\boldsymbol{\mu}}$: simple local system on $\mathcal{Y}^0_{m(\boldsymbol{\mu})}$.

For $0 \le m \le n$

$$\widetilde{\mathcal{X}}_m = \{(x, v, gB^\theta) \in G^{\iota\theta} \times V \times K/B^\theta \mid (g^{-1}xg, g^{-1}v) \in B^{\iota\theta} \times M_m\}$$

$$\mathcal{X}_m = \bigcup_{g \in K} g(B^{\iota\theta} \times M_m)$$

$$\pi : \widetilde{\mathcal{X}}_n \to \mathcal{X}_n = G^{\iota\theta} \times V, \quad (x, v, gB^{\iota\theta}) \mapsto (x, v)$$

$\mathcal{Y}_m$ is open dense in $\mathcal{X}_m$, $\pi_*\bar{\mathbb{Q}}_l|_{\mathcal{X}_{\text{uni}}} \simeq (\pi_1)_*\bar{\mathbb{Q}}_l$

Proposition 2

1. $\pi_*\bar{\mathbb{Q}}_l$ is equipped with W_n action, is decomposed as

$$\pi_*\bar{\mathbb{Q}}_l[\dim \mathcal{X}_n] \simeq \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} V_{\boldsymbol{\mu}} \otimes \mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\mu})}, \mathcal{L}_{\boldsymbol{\mu}})[\dim \mathcal{X}_{m(\boldsymbol{\mu})}]$$

2. $\mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\mu})}, \mathcal{L}_{\boldsymbol{\mu}})|_{\mathcal{X}_{\text{uni}}} \simeq \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\mu}^\bullet}, \bar{\mathbb{Q}}_l)[a]$ for some $\boldsymbol{\mu}^\bullet \in \mathcal{P}_{n,2}$, where $a = \dim \mathcal{O}_{\boldsymbol{\mu}^\bullet} - \dim \mathcal{X}_{\text{uni}} - \dim \mathcal{X}_{m(\boldsymbol{\mu})} + \dim \mathcal{X}_n$.

Theorem 2 (explicit correspondence)

Under the notation of Theorem 1, we have $\boldsymbol{\mu}^\bullet = \boldsymbol{\mu}$. Hence the Springer correspondence is given by $\mathcal{O}_{\boldsymbol{\mu}} \leftrightarrow V_{\boldsymbol{\mu}}$.

Theorem 2 was proved by Kato. Our proof uses "restriction thereom".

$P = LU_P$: θ-stabel parabolic subgroup of G s.t. $L^\theta \simeq GL_1 \times Sp_{2n-2}$.
$V_1 \oplus V' \subset V$, $GL_1 = GL(V_1)$, $Sp_{2n-2} = Sp(V')$.

For $z = (x, v) \in G^{\iota\theta}_{\text{uni}} \times V$, $z' = (x', v') \in L^{\iota\theta}_{\text{uni}} \times V'$,

$$Y_{z,z'} = \{g \in K \mid g^{-1}xg \in x'U_P^{\iota\theta}, g^{-1}v \in v' + V_1\}$$

Put $d_{z,z'} = (\dim Z_K(z) - \dim Z_{L^\theta}(z'))/2 + \dim U_P^\theta$

Restriction Theorem

Let ρ_z^G irred. rep. of W_n corresp. to $z \in \mathcal{O}$, $\rho_{z'}^L$ irred. rep. of W_{n-1} corresp. to $z' \in \mathcal{O}'$. Then $\langle \rho_Z^G, \rho_{z'}^L \rangle_{W_{n-1}}$ coincides with the number of irreducible components of $Y_{z,z'}$ with dimension $d_{z,z'}$

Kostka polynomials and exotic nilcone

For $\mathcal{O}_{\boldsymbol{\lambda}} \subset \mathfrak{g}^{-\theta}_{\text{uni}} \times V$, let $K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbb{Q}}_l)$.

Conjecture (Achar-Henderson)

- $\mathcal{H}^i K = 0$ unless $i \equiv 0 \pmod 4$.
- For $(x, v) \in \mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$, $\quad t^{a(\boldsymbol{\lambda})} \sum_i \dim(\mathcal{H}^{4i}_{(x,v)} K) t^{2i} = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$

W_n acts on $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n] \supset V_{\boldsymbol{\lambda}}$: Specht module

Define $R^{\boldsymbol{\lambda}} = \mathbb{C}[x]/I_{\boldsymbol{\lambda}}$, where $I_{\boldsymbol{\lambda}} = \{P \in \mathbb{C}[x] \mid P(\partial) f = 0 \ \forall f \in V_{\boldsymbol{\lambda}}\}$

$R^{\boldsymbol{\lambda}} = \bigoplus_{i \geq 0} R^{\boldsymbol{\lambda}}_i$: Graded W_n-module

Conjectue (S) $\quad \sum_{i \geq 0} \langle R^{\boldsymbol{\lambda}}_i, \chi^{\boldsymbol{\mu}} \rangle t^i = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$

Remark Recently Kato proved Conjecture (S)
Conjecture (S) + his another result $\Longrightarrow$ Conjecture (AH)

Future Problem

- Discuss the case for $G^{\iota\theta} \times V$ with $\mathrm{ch}\,\mathbb{F}_q = 2$. Known by Kato that there exists an interesting relationship with Springer correpondence for symplectic groups with even characteristic.
- Extension to the case $G^{\iota\theta} \times V^{r-1}$ for $r \geq 2$, and discuss the relationship with Kostka functions associated to complex reflection groups.
- Extension to the general symmetric space.
  $\theta : G \to G$ involution, $K = G^\theta$. Consider the variety V with K action such that the number of K-orbits on $G^{\iota\theta} \times V$ is finite. Develope the theory of character sheaves on $G^{\iota\theta} \times V$.