Logic, Automata and Relations

- **A logical formula ϕ with a single variable x represents a set L**

  $w \in L$ **iff** $w \models \phi$

- **Ex.: $P(x)$ be a formula, which is satisfied if x is an even integer**

  $2 \models P(x)$

  $P(x)$ **can be seen as the set of even numbers**

- **Formula with n variables $x_1, \ldots, x_n$ represents n-ary relation**

  $(w_1, \ldots, w_n) \in R$ **iff** $w_1, \ldots, w_n \models \phi$

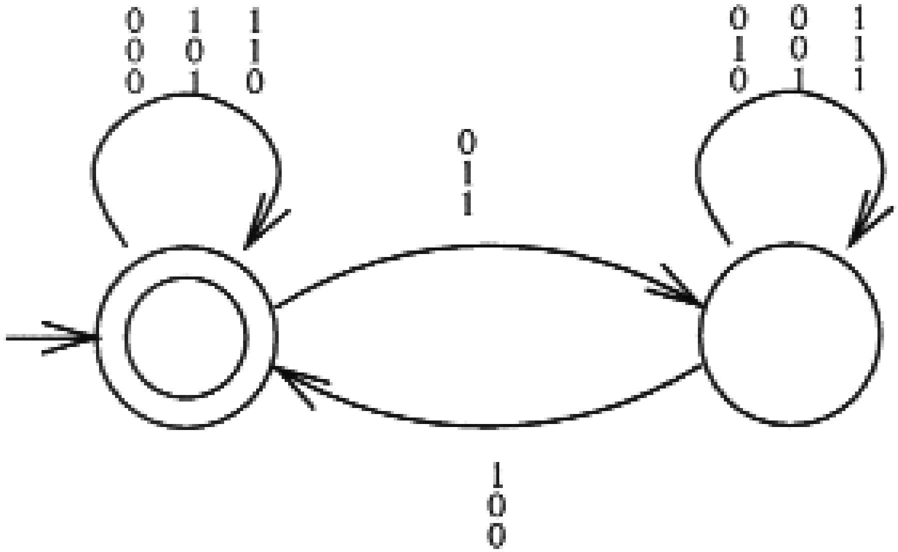
- **A (tree) automaton can represents a relation on strings (trees)**
- **String that represents a duple $(aba, \varepsilon, bbba)$:**

$$[aba, \varepsilon, bbba] = \begin{matrix} a \\ \perp \\ b \end{matrix} \quad \begin{matrix} b \\ \perp \\ b \end{matrix} \quad \begin{matrix} a \\ \perp \\ b \end{matrix} \quad \begin{matrix} \perp \\ \perp \\ a \end{matrix}$$

- **Ex.: Automata that represents addition relation R_+ on binary representation**
  - **R_+ is defined by**

    $(x^R, y^R, z^R) \in R_+$ **iff** $x = y + z$
  - **Automata on alphabet $\left\{ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}, \cdots, \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right\}$ that recognizes the relation R_+ (shown in the following page)**

• **Ex. (cont.):**



- $1100 = 0101 + 0111$ **on binary. Thus,**

$$[0011, 1010, 1110] = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}$$ **is accepted**

Binary Relation defined by Tree Automata

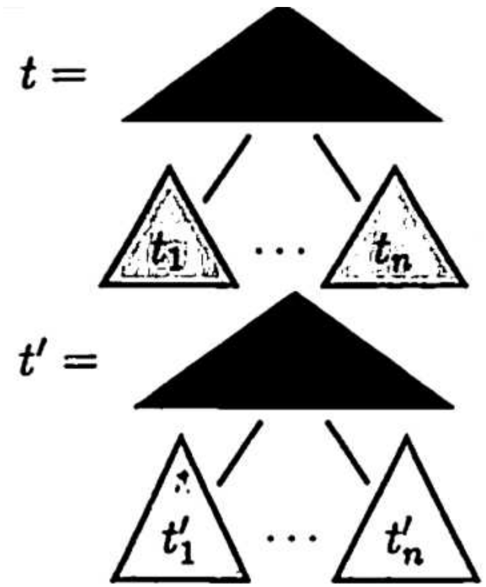
- **Class Rec$_\times$: Relation $L_1 \times L_2$ for regular tree languages L_1 and L_2**
  - $\Delta = \{(t,t) \mid t \in \mathsf{T}(\mathcal{F})\}$ **is not in** Rec$_\times$
- **Class Rec : Relation $\{(t,u) \mid [t,u] \in L\}$ for regular tree language L**
  - **Tuple of trees (for the case $n \geq m$)**

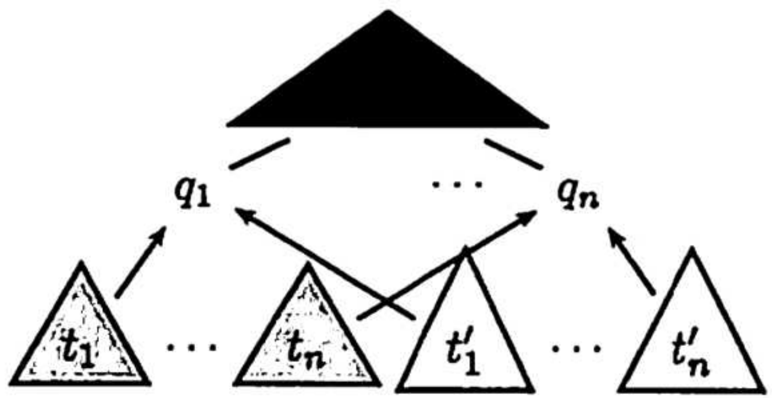
$$[f_1(t_1,\ldots,t_n), f_2(u_1,\ldots,u_m)]$$
$$= f_1f_2([t_1,u_1],\ldots,[t_m,u_m],[t_{m+1},\perp],\ldots,[t_n,\perp])$$

  - **Ex.:** $[f(g(a),g(a)), f(f(a,a),a)]$

$$= ff(gf(aa,\perp a), ga(a\perp))$$

- **Class GTT: Relation fixed from NFTAs A_1 A_2 as follows:**
  Let $A_i = (Q_i, \mathcal{F}, \emptyset, \Delta_i)$
  $C[t_1, \ldots, t_n] \, R \, C[u_1, \ldots, u_n]) \iff$
  for some C **and** $q_j \in Q_1 \cap Q_2$**,**
  $t_j \rightarrow^*_{A_1} q_j$ **and** $u_j \rightarrow^*_{A_2} q_j$





- **Ex. relation** R **in** Rec: $\mathcal{F} = \{a, \Omega, g(), f(,)\}$

  $tRu \overset{\text{def}}{\iff} u \in (\{t\} \cdot_\Omega \mathsf{T}(\mathcal{F}))$

  NFTA A **with** $Q^f = \{q'\}$ **that accepts** $[t, u]$

  $aa \to q'$, $gg(q') \to q'$, $ff(q', q') \to q'$,
  $\Omega a \to q'$, $\Omega g(q) \to q'$, $\Omega f(q, q) \to q'$, $\Omega\Omega \to q'$
  $\perp a \to q$, $\perp g(q) \to q$, $\perp f(q, q) \to q$, $\perp \Omega \to q$

  Acception ex.:

  for $t = f(g(\Omega), g(\Omega))$ **and** $u = f(g(g(a)), g(\Omega))$**,**

  $$
  \begin{aligned}
  [tu] &= ff(gg(\Omega g(\perp a)), gg(\Omega\Omega)) \\
  &\to_A^* ff(gg(\Omega g(q)), gg(q')) \\
  &\to_A^* ff(gg(q'), q') \\
  &\to_A ff(q', q') \\
  &\to_A q'
  \end{aligned}
  $$

- **Ex. relation** R^* **in** GTT $R^* : \mathcal{F} = \{\times, +, 0, 1\}$

  $tRu \overset{\text{def}}{\iff} \exists C, t' \ \ t = C[0 \times t'] \land u = C[0]$

  GTT **by** A_1, A_2 **that defines** R^*

  A_1**:** $0 \to q \quad\quad 0 \to q_0 \quad\quad 1 \to q$
  $\quad\quad q + q \to q \quad q \times q \to q \quad q_0 \times q \to q_0$

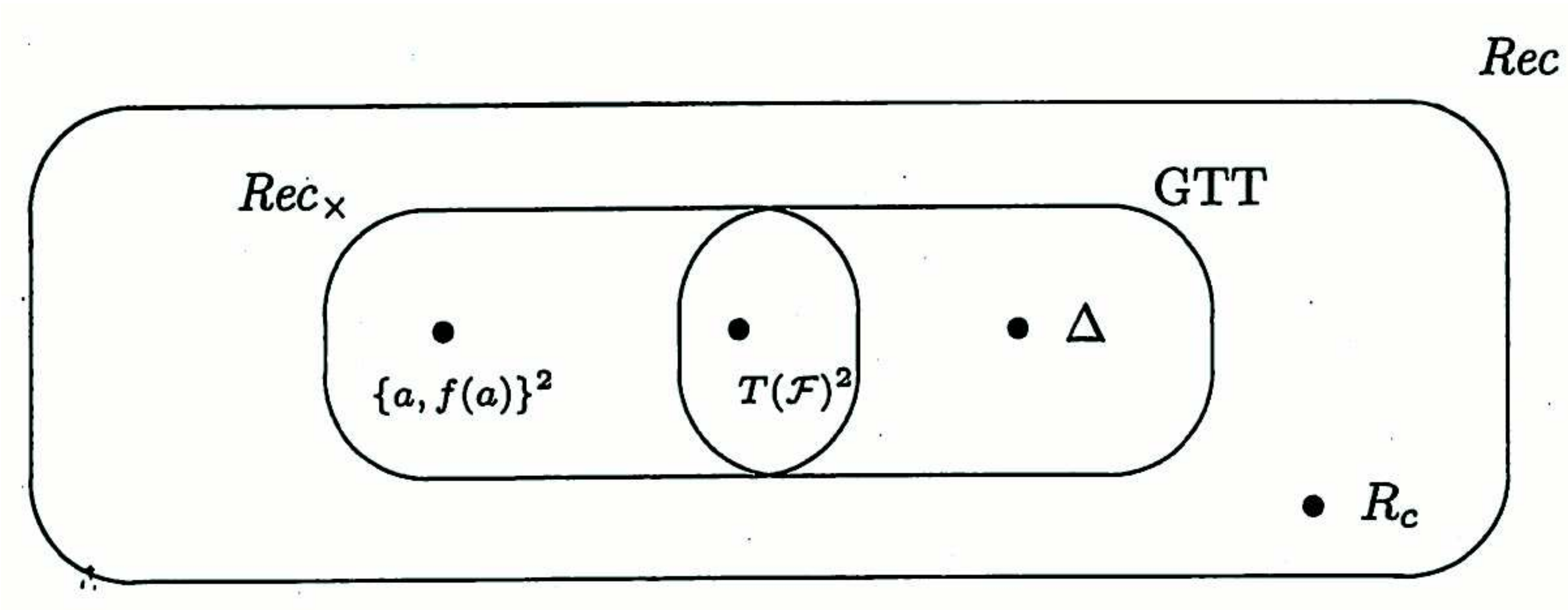
  A_2**:** $0 \to q_0$

  Acception ex.:

  for $t = 1 + ((0 \times 1) \times 1$ **and** $u = 1 + 0$**,**

  $t \to^*_{A_1} 1 + ((q_0 \times q) \times q)$
  $\to_{A_1} 1 + (q_0 \times q)$
  $\to_{A_1} 1 + q_0$
  $u \to_{A_2} 1 + q_0$

- **Relationship among classes**



Closure property of Rec

- **Inherits from NFTA (union, intersection, etc)**
- i**'th projection** $R_i \subseteq T^{n-1}$ **of** $R \subseteq R^n$**:**

$$R_i(t_1, \ldots, , t_{i-1}, t_{i+1}, \ldots, t_n)$$
$$\stackrel{\text{def}}{\iff} \exists t \;\; R(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$$

- **E.:** $R = \{(a, a, a), (a, a, c), (a, b, c)\}$

$R_2 = \{(a, a), (a, c)\}$

- i**'th cylindrification** $R^i \subseteq T^{n+1}$ **of** $R \subseteq R^n$**:**

$$R^i(t_1, \ldots, t_{n+1})$$
$$\stackrel{\text{def}}{\iff} \;\; R(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+1})$$

- **Ex.:** $R = \{(a, a), (a, c)\} \subseteq \{a, b, c\}^2$

$R^2 = \{(a, a, a), (a, b, a), (a, c, a), (a, a, c), (a, b, c), (a, c, c)\}$

- Rec **is closed under projection and cylindrification**
- **Proof: Projection** R_i **is given as a linear tree homomorphism** h **from** R**:**

  $$h_{\mathcal{F}}(f_1 \cdots f_n(x_1, \ldots, x_k))$$
  $$= f_1 \cdots f_{i-1} f_{i+1} \cdots f_n(x_1, \ldots, x_{k'}),$$

  (arity$(f_1 \cdots f_n) = k \geq k' =$ arity$(f_1 \cdots f_{i-1} f_{i+1} \cdots f_n)$**)**
  Cylindrification R^i **is given the inverse image of** h

Closure property of GTT

- GTT **is closed under transitive closure**
- **Proof sketch: (augmenting** ε**-rules)**

  for states q, q' **in both** A_1 **and** A_2 **such that**

  $\exists t \ t \to^*_{A_1} q \wedge t \to^*_{A_2} q'$

  if $q \not\to^*_{A_2} q'$**,** $q \to q'$ **is added to** A_2

  if $q' \not\to^*_{A_1} q$**,** $q' \to q$ **is added to** A_1

- **Proof sketch (cont.)**