NFTA and Regular Tree Expression

- **Prop.: A regular tree expression exists which is equivalent to a given DFTA** $A = (Q, \mathcal{F}, Q^f, \Delta)$
- **Proof: Let** $Q = \{1, \ldots, n\}$
  - **For a set** K **(**$\subseteq Q$**), introduce** $R^{(m)}_{K,j}$ **to represent the regular tree expressions that denote** $t \in \mathsf{T}(\mathcal{F} \cup K)$ **such that** $t \to^*_A j$ **without use of states greater than** m
  - 例：Δ : $a \to 1, f(1,1) \to 2, f(1,2) \to 2,\quad Q^f = \{2\}$

    $R^{(0)}_{\{1\},2}$: $\{f(1,1)\}$,

    $R^{(1)}_{\{1\},2}$: $R^{(0)}_{\{1\},2} \cup \{f(1,a), f(a,1), f(a,a)\}$,

    $R^{(2)}_{\{1\},2}$: $R^{(1)}_{\{1\},2} \cup \{f(1, f(1,1)), f(a, f(1,1)), f(1, f(a,1)), \ldots\}$

- **Proof (Cont.):**
  - The target expression is given as the union of $R^{(n)}_{\emptyset j}$ for all $j \in Q^f$, where $Q = \{1, \ldots, n\}$
  - Recursive definition of $R^{(m)}_{K,j}$

    Basis: ($m = 0$)

    Let $a_1, \ldots, a_p$ be a's satisfying $a \to_A j$, and $t_1, \ldots, t_\ell$ be t's satisfying $t = f(q_1, \ldots, q_k) \to_A j$ for some $q_i \in K$.

    If $j \in K$,

    $$R^{(0)}_{K,j} = j + a_1 + \cdots + a_p + t_1 + \cdots + t_\ell$$

    If $j \notin K$,

    $$R^{(0)}_{K,j} = a_1 + \cdots + a_p + t_1 + \cdots + t_\ell$$

- **Proof (cont.):**
  - **Recursive definition $R_{K,j}^{(m)}$ (cont.)**

    Ind.: ($m > 0$)

    $$R_{K,j}^{(m)} =$$
    $$R_{S,j}^{(m-1)} + R_{K\cup\{m\},j}^{(m-1)} \cdot_m \left(R_{K\cup\{m\},m}^{(m-1)}\right)^{*,m} \cdot_m R_{K,m}^{(m-1)}$$

- **Ex.:** $\Delta : a \to 1, f(1,1) \to 2, f(1,2) \to 2,\quad Q^f = \{2\}$
  - **Calculation of** $R^{(0)}_{K,j}$

$j \backslash K$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
1	a	$1+a$	a	$1+a$
2	$\emptyset$	$f(1,1)$	2	$2+f(1,1)+f(1,2)$

  - **Calculation of** $R^{(1)}_{K,j}$

$$R^{(1)}_{\emptyset,2} = R^{(0)}_{\emptyset,2} + R^{(0)}_{\{1\},2} \cdot_1 \left(R^{(0)}_{\{1\},1}\right)^{*,1} \cdot_1 R^{(0)}_{\emptyset,1}$$
$$= \emptyset + f(1,1) \cdot_1 (1+a)^{*,1} \cdot_1 a$$
$$= f(1,1) \cdot_1 (1+a) \cdot_1 a = f(a,a)$$
$$R^{(1)}_{\{2\},2} = R^{(0)}_{\{2\},2} + R^{(0)}_{\{1,2\},2} \cdot_1 \left(R^{(0)}_{\{1,2\},1}\right)^{*,1} \cdot_1 R^{(0)}_{\{2\},1}$$
$$= 2 + (2 + f(1,1) + f(1,2)) \cdot_1 (1+a)^{*,1} \cdot_1 a$$
$$= 2 + (2 + f(1,1) + f(1,2)) \cdot_1 (1+a) \cdot_1 a$$
$$= 2 + f(a,a) + f(a,2)$$

- **Ex. (cont.):**
  - $R^{(1)}_{\emptyset,2} = f(a,a) \quad R^{(1)}_{\{2\},2} = 2 + f(a,a) + f(a,2)$
  - **Calculation of** $R^{(2)}_{\emptyset,2}$

$$
\begin{aligned}
R^{(2)}_{\emptyset,2} &= R^{(1)}_{\emptyset,2} + R^{(1)}_{\{2\},2} \cdot_2 \left(R^{(1)}_{\{2\},2}\right)^{*,2} \cdot_2 R^{(1)}_{\emptyset,2} \\
&= f(a,a) + (2 + f(a,a) + f(a,2)) \\
&\qquad \cdot_2 (2 + f(a,a) + f(a,2))^{*,2} \cdot_2 f(a,a) \\
&= f(a,a) + (2 + f(a,a) + f(a,2))^{*,2} \cdot_2 f(a,a) \\
&\quad + f(a,2) \cdot_2 (2 + f(a,a) + f(a,2))^{*,2} \cdot_2 f(a,a) \\
&= f(a,a) + f(a,2)^{*,2} \cdot_2 f(a,a) \\
&\quad + f(a,2) \cdot_2 f(a,2)^{*,2} \cdot_2 f(a,a) \\
&= f(a,a) + f(a,2)^{*,2} \cdot_2 f(a,a) = f(a,2)^{*,2} \cdot_2 f(a,a)
\end{aligned}
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- **Theorem: A tree language is regular, if and only if it can be denoted by a regular tree expression**
- **Proof: ($\subseteq$): By the previous Prop.
($\supseteq$): By induction on the structure of regular tree expression**