Minimization

- Relation $\equiv\subseteq S \times S$ is an equivalent relation if the following properties hold:
  - Reflectively: $\forall t \in S,\ t \equiv t$
  - Symmetricity: $\forall t, s \in S,\ t \equiv s \Rightarrow s \equiv t$
  - Transitivity: $\forall t, s, u \in S,\ (t \equiv s \wedge s \equiv u) \Rightarrow t \equiv u$
- S is divided into equivalence classes by $\equiv$. Index $|\equiv|$ is the number of the classes
  - Equivalence class containing t:

$$[t]_{\equiv} = \{s \mid t \equiv s\}$$

  - $S = \bigcup_{t \in S} [t]_{\equiv}$

- Equivalence relation $\equiv$ on $\mathsf{T}(\mathcal{F})$ is congruence relation if the following property holds
  $t_1 \equiv s_1, \ldots, t_n \equiv s_n$ implies $\forall f \in \mathcal{F}$, $f(t_1, \ldots, t_n) \equiv f(s_1, \ldots, s_n)$
- Congruent relation $\equiv_L$ on $\mathsf{T}(\mathcal{F})$ is determined by a language L:
  $t \equiv_L s$ if and only if
  $C[t] \in L \iff C[s] \in L$ for any context C
- Congruent relation $\equiv_A$ on $\mathsf{T}(\mathcal{F})$ is determined by a complete DFTA $A = (Q, \mathcal{F}, Q^f, \Delta)$:
  $t \equiv_A s$ if and only if $t \rightarrow_A^* q$ and $s \rightarrow_A^* q$

- **Thorem(Myhill-Nerode)**
  The following conditions are equivalent
  1. L **is regular**
  2. L **is equal to union of some equivalence classes of a congruence relation with finite index**
  3. $\equiv_L$ **has a finite index**

- **Proof of ($1 \Rightarrow 2$): Let** $A = (Q, \mathcal{F}, Q^f, \Delta)$ **be a complete DFTA recognizing** L**. Then**
  - $\equiv_A$ **has a finite index since index is not greater than number of states of** A**, and**
  - $L = \bigcup_{t \in L} [t]_{\equiv_A}$
- **Proof of ($2 \Rightarrow 3$): Letting** $\sim$ **be congruence relation determined by 2, we show that**

  $\forall t \in \mathsf{T}(\mathcal{F})$, $[t]_\sim \subseteq [t]_{\equiv_L}$, **i.e.** $| \sim | \geq | \equiv_L |$
  - **Letting** $s \in [t]_\sim$, $s \sim t$. **Since** $\sim$ **is congruent,** $C[s] \sim C[t]$ **for any context** C**. Thus** $C[s] \in L \iff C[t] \in L$ **holds by 2. Therefore** $s \equiv_L t$**, i.e.** $s \in [t]_{\equiv_L}$

- **Proof of (3 $\Rightarrow$ 1) : Construct FTA** $A_{\mathsf{min}} = (Q, \mathcal{F}, Q^f, \Delta)$ **from** $\equiv_L$**:**
  - $Q = \{[t]_{\equiv_L} \mid t \in \mathsf{T}(\mathcal{F})\}$
  - $f([t_1]_{\equiv_L}, \ldots, [t_n]_{\equiv_L}) \to [f(t_1, \ldots, t_n)]_{\equiv_L} \in \Delta$
    (DFTA from congruence property of $\equiv_L$**)**
  - $Q^f = \{[t]_{\equiv_L} \mid t \in L\}$
- **From** $t \to^*_{A_{\mathsf{min}}} [t]_{\equiv_L}$ **and construction of** Q^f**,** A_{min} **recognizes** L

- **Cor.: Minimum DFTA recognizing a regular tree language L is uniquely determined as A_{min} in Myhill-Nerode theorem under renaming**
- **Proof: Considering DFTA A such that $L = L(A)$, from Myhill-Nerode theorem $[t]_{\equiv_A} \subseteq [t]_{\equiv_L}$ for any t**
  - **States of A are not less than states of A_{min}. Thus A_{min} is minimum**
  - **Uniqueness is clear from $[t]_{\equiv_A} \subseteq [t]_{\equiv_L}$**
- **Classes of $\equiv_A$ (sates of A) are indistinguishable if they are contained a class of $\equiv_L$.**

- **Minimization of DFA A: merging indistinguishable classes of $\equiv_A$ (states of A)**
- **Property to distinguish classes**
  - **If $q \in Q^f$ and $q' \in Q \setminus Q^f$, then q and q' are distinguishable**
  - **If q and q' are distinguishable, and**
    $f(\cdots p \cdots) \to q$, $f(\cdots p' \cdots) \to q' \in \Delta$
    then p and p' are distinguishable

- Ex.: $A = (\{q_a, q_b, q_{ab}, q_{ba}\}, \{a, b, f(,)\}, \Delta, \{q_{ab}, q_{ba}\})$

$a \to q_a$,

$b \to q_b$,

f:

	q_a	q_b	q_{ab}	q_{ba}
q_a	q_a	q_{ab}	q_{ab}	q_{ab}
q_b	q_{ba}	q_b	q_{ba}	q_{ba}
q_{ab}	q_{ab}	q_{ab}	q_{ab}	q_{ab}
q_{ba}	q_{ba}	q_{ba}	q_{ba}	q_{ba}

- Enumerate distinguishable states
  - every final state and non-final state are distinguishable
  - From $f(q_a, q_a) \to q_a$ and $f(q_b, q_a) \to q_{ba}$, q_a and q_b are distinguishable

	q_a	q_b	q_{ab}	q_{ba}
q_b	×	—	—	—
q_{ab}	×	×	—	—
q_{ba}	×	×		—

×: distinguishable -: N/A

  - q_{ab} and q_{ba} can be merged

- **Decidable problems**
  - **Emptiness:** $L(A) = \emptyset$ **?**
  - **Finiteness:** $L(A)$ **is finite ?**

    Existence of loop $C[q] \rightarrow_A^* q$ **?**
  - **Singleton set property:** $L(A)$ **is singleton ?**
  - **Equivalence:** $L(A) = L(A')$ **?**