AWCBB

- **AWCBB (Automata with constraints Between Brothers):**
  - AWE DC in which constraints are restricted to $i = j$, $i \neq j$ $(i, j \in N)$
- Ex.:
  - $A = (\{q\}, \mathcal{F}, \{q\}, \Delta)$

    $\mathcal{F} = \{f(,), a\}$

    $\Delta = \{a \to q,\ f(q, q) \overset{1=2}{\to} q\}$
  - $L(A)$ is the set of complete binary trees
- AWCBB is closed by union, intersection, and complement

- Emptiness problem for AWCBB is decidable
- Key of the proof:
  - Consider a deterministic AWCBB. Then if $q_1 \neq q_2$ the following rule never be used

    $$f(q_1, q_2) \stackrel{1=2}{\rightarrow} q$$
  - For the following rule, we must know whether more than one trees are reachable to the given state

    $$a \rightarrow q,\ b \rightarrow q,\ f(q, q) \stackrel{1\neq 2}{\rightarrow} q'$$
- $M_{\mathcal{F}}$: maximum number of arguments of symbols in $\mathcal{F}$
- $L(q) \stackrel{\text{def}}{\Longleftrightarrow} \{t \mid t \rightarrow_A^* q\}$

- **Lemma: for a rule $f(q_1, \ldots, q_n) \xrightarrow{c} q$ of deterministic AWCBB, $|L(q)| \geq M_{\mathcal{F}}$ if**
  $|L(q_1)| \geq 1, \ldots, |L(q_n)| \geq 1$, $\exists q_i . |L(q_i)| \geq M_{\mathcal{F}}$,
  and $\exists t_1 \in L(q_1), \ldots, t_n \in L(q_n).\ f(t_1, \ldots, t_n) \models c$
- **Proof sketch: let $t = f(t_1, \ldots, t_n)$. Note that c has no negation**
  - Constraint $i = j$ satisfied by t is also satisfied by a term obtained from t by replacing t_i with a term in $|L(q_i)|$, since $q_i = q_j$
  - Constraint $i \neq j$ satisfied by t is also satisfied by a term obtained from t by replacing t_i with an appropriate term in $|L(q_i)|$ in either cases that $q_i \neq q_j$ or $q_i = q_j$

- **Th.: Emptiness for an AWCBB is decidable**
- **Proof sketch: Assume deterministic AWCBB with rule set Δ**
  - **Initialize $L_p := \emptyset$, and repeat the following step for each state p until L_p's saturate**
  - $L_q := L_q \cup \{t\}$ **for t such that $t \notin L_q$, $|L_q| \leq M_{\mathcal{F}}$ and the following holds:**

    $$f(q_1, \ldots, q_n) \xrightarrow{c} q \in \Delta, \quad t_1 \in L_{q_1}, \ldots, t_n \in L_{q_n},$$

    and

    $$t = f(t_1, \ldots, t_n),\ t \models c$$
  - **It is empty if $L_q = \emptyset$ for all accepting states q**

Reduction automata

- **AWEDC that satisfies the conditions**
  - **States are ordered $<$, and**
  - **For any $f(q_1, \ldots, q_n) \xrightarrow{c} q$,**

    $\forall i. q_i > q$ **if c contains EQUALITY**

    $\forall i. q_i \geq q$ **otherwise**
- **Ex.: Deterministic and complete reduction automaton accepting terms containing $g(g(t,s),t)$ for some s,t)**

  $a \rightarrow q_\top,\ g(q_\top, q_\top) \rightarrow q_g,\ g(q_\top, q_g) \rightarrow q_g,\ g(q_g, q_\top) \xrightarrow{11=2} q_f,$
  $g(q_g, q_\top) \xrightarrow{11\neq 2} q_g,\ g(q_g, q_g) \xrightarrow{11=2} q_f,\ g(q_g, q_g) \xrightarrow{11\neq 2} q_g,$
  $g(q, q_f) \rightarrow q_f,\ g(q_f, q) \rightarrow q_f$ **where** $q \in \{q_\top, q_g, q_f\}$

Property of reduction automata

- Closed under union and intersection
  Open for complement
- Emptiness problem
  - Decidable if complete and deterministic
  - Undecidable if non-deterministic
- Finiteness problem (finiteness of $L(A)$) is decidable