NFTA with constraints

- **(Dis)-equality constraint:** $\pi = \pi'$, $\pi \neq \pi'$ where $\pi, \pi' \in N^*$
  - For labelled tree t,

$$t \models \pi = \pi' \quad \stackrel{\text{def}}{\iff} \quad \pi, \pi' \in \text{Pos}(t) \wedge t|_\pi = t|_{\pi'}$$

- **AWEDC** (Automata with equality and disequality constraints):
  - NFTA with transition rules in forms of

$$f(q_1, \ldots, q_n) \xrightarrow{c} q$$

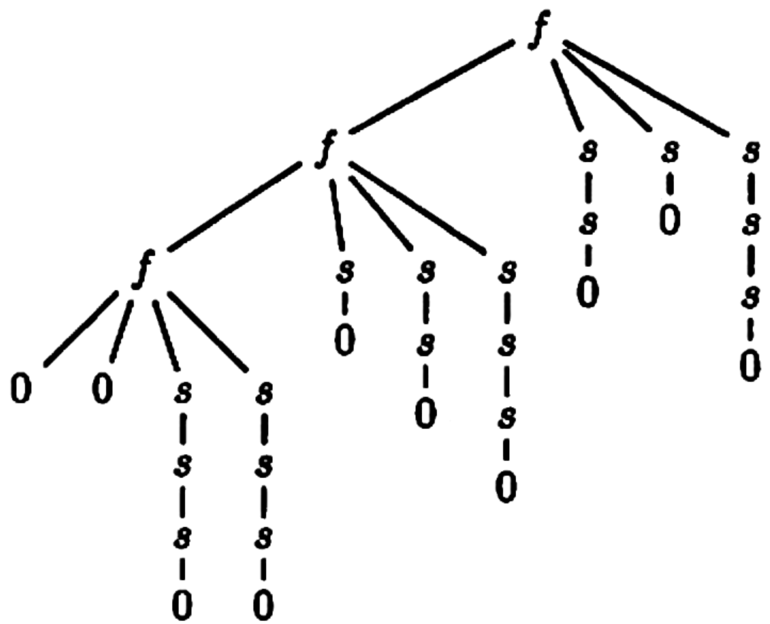
  where c is a formula combined with (dis)-equality constraints, $\vee$, and $\wedge$

- **Transition relation $\to_A$ of AWEDC $A = (Q, \mathcal{F}, Q^f, \Delta$**
  For $f(q_1, \ldots, q_n) \xrightarrow{c} q \in \Delta$**,**
  $C[t] \to_A^* C[f(q_1, \ldots, q_n)]$**,** $t \models c$
  $C[t] \to_A^* C[q]$

- **Ex.:**
  - $A = (\{q\}, \{f(,), a\}, \{q\}, \Delta)$
    $\Delta = \{a \to q,\ f(q, q) \xrightarrow{1=2} q\}$
  - $f(f(a, a), f(a, a))$ **is accepted, and** $f(a, f(a, a))$ **is not accepted**

- **Ex: an AWEDC that accepts the verification trees for the addition of natural numbers**
  - $A = (\{q_0 . q_n, q_f\}, \{0, s(), f(,)\}, \{q_f\}, \Delta)$

$$0 \rightarrow q_0, \quad s(q_0) \rightarrow q_n, \quad s(q_n) \rightarrow q_n,$$
$$f(q_0, q_0, q_0, q_0) \rightarrow q_f$$
$$f(q_0, q_0, q_n, q_n) \xrightarrow{3=4} q_f \quad f(q_0, q_n, q_0, q_n) \xrightarrow{2=4} q_f$$
$$f(q_f, q_n, q_n, q_n) \xrightarrow{14=4 \wedge 21=12 \wedge 131=3} q_f$$



Properties on AWEDC

- **Completion:**
  - Ex.: for $a \to q,\ f(q,q) \xrightarrow{1=2} q$, enough to add

    $f(q,q) \xrightarrow{1\neq 2} q_\perp$,

    $f(q_\perp,q) \to q_\perp$, $f(q,q_\perp) \to q_\perp$, $f(q_\perp,q_\perp) \to q_\perp$
- **Determination:** (possible in preserving completeness)
  - Idea: for $a \xrightarrow{c_1} q_1 \quad a \xrightarrow{c_2} q_2$,

    produce rules

    $a \xrightarrow{c_1 \wedge c_2} \{q_1, q_2\} \quad a \xrightarrow{c_1 \wedge \neg c_2} \{q_1\}$

    $a \xrightarrow{\neg c_1 \wedge c_2} \{q_2\} \quad a \xrightarrow{\neg c_1 \wedge \neg c_2} \{\}$

- **Decidable problem**
  - **Membership:** $w \in L(A)$ **?**
- **Undecidable problems**
  - **Emptiness:** $L(A) = \emptyset$ **?**

    Prove by reducing the Post Correspondence Problem (PCP), know as undecidable

- **Post Correspondence Problem (PCP)**
  - **Input:** $P \subseteq \{a,b\}^* \times \{a,b\}^*$
  - **Solution of P:** $w_1 \cdots w_k = w'_1 \cdots w'_k$,
    where $k > 0$, and $(w_i, w'_i) \in P$ for each i
  - **PCP: Is there a solution for input P ?**
- **Ex:** $P = \left\{ \begin{pmatrix} abb \\ a \end{pmatrix}, \begin{pmatrix} b \\ bb \end{pmatrix} \right\}$

  A solution of P is $abbbbbb$, where $\begin{pmatrix} abb \\ a \end{pmatrix}\begin{pmatrix} b \\ bb \end{pmatrix}\begin{pmatrix} b \\ bb \end{pmatrix}$

- **Proof for undecidability of emptiness problem of AWEDC**
  - $w(t)$: $a_1(\cdots a_n(t)\cdots)$ **if** $w = a_1 \cdots a_n$
  - **Let** $\mathcal{F} = \{a(), b(), 0, h(,,)\}$
  - **For each** $(w, w') \in P$**, construct NFTA s.t.**

    $w(q_0) \rightarrow^* q_w$, $w'(q_0) \rightarrow^* q_{w'}$,

    where rules on q_0 **are the following:**

    $a(q_0) \rightarrow q_a$, $b(q_0) \rightarrow q_b$

    Rules for the ex.:

    $a(q_0) \rightarrow q_a$, $b(q_0) \rightarrow q_b$,

    $b(q_b) \rightarrow q_{bb}$, $a(q_{bb}) \rightarrow q_{abb}$

- **Proof for undecidabillity (cont.)**
  - **Add states q, q_f. Construct rules**

$$0 \to q_0,\ h(q_0, q_0, q_0) \to q$$

For each $(w, w') \in P$

$$a(q_w) \to q_a,\ b(q_w) \to q_b,\ a(q_{w'}) \to q_a,\ b(q_{w'}) \to q_b,$$

$$h(q_w, q, q_{w'}) \xrightarrow{11^{|w|}=21 \wedge 31^{|w'|}=23} q$$

$$h(q_w, q, q_{w'}) \xrightarrow{11^{|w|}=21 \wedge 31^{|w'|}=23 \wedge 1=3} q_f$$

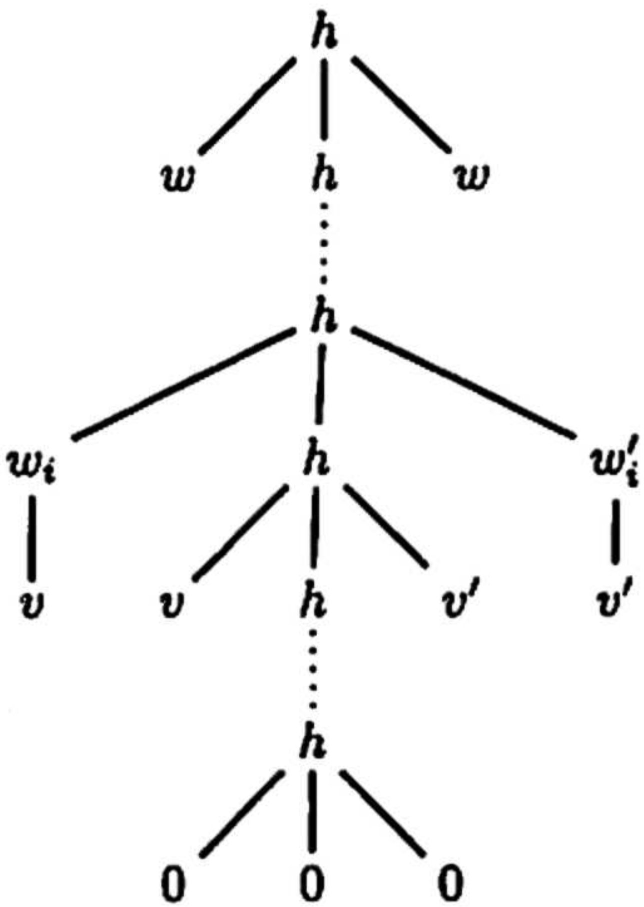
Ex.: for $(b, bb) \in P$,

$$a(q_b) \to q_a,\ b(q_b) \to q_b,\ a(q_{bb}) \to q_a,\ b(q_{bb}) \to q_b,$$

$$h(q_b, q, q_{bb}) \xrightarrow{11=21 \wedge 311=23} q$$

$$h(q_b, q, q_{bb}) \xrightarrow{11=21 \wedge 311=23 \wedge 1=3} q_f$$

- **Proof for undecidabillity (cont.)**



- Q10: Transform the PCP in page 6 as an example into an AWEDC according to the proof of undecidability of AWEDC. Moreover, give the tree corresponding to the solution in page 6