4 Unitary representation

In this section, we consider the following problem:

Question 4.1. *Which irreducible representations in Theorem 3.27 are unitary?*

4.1 Inner product

Definition 4.2. Let W be a vector space over $\mathbb{C}$. A map $\langle\ ,\ \rangle : W \times W \longrightarrow \mathbb{C}$ is a *Hermitian form* if

$$\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle \text{ for } \forall u_1, u_2, v \in W,$$

$$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \text{ for } \forall u, v \in W \text{ and } \forall \alpha \in \mathbb{C},$$

$$\langle u, v_1 + v_2\rangle = \langle u, v_2\rangle + \langle u, v_2\rangle \text{ for } \forall u, v_1 v_2 \in W,$$

$$\langle u, \alpha v\rangle = \overline{\alpha}\langle u, v\rangle \text{ for } \forall u, v \in W \text{ and } \forall \alpha \in \mathbb{C},$$

$$\langle u, v\rangle = \overline{\langle v, u\rangle} \text{ for } \forall u, v \in W.$$

A hermitian form $\langle\ ,\ \rangle$ is called

- *non-degenerate* if for $W \ni \forall u \neq 0$, there exists $v \in W$ such that $\langle u, v\rangle \neq 0$,
- *positive definite* if for $W \ni \forall u \neq 0$, $\langle u, u\rangle > 0$.

Note that positive definite $\Rightarrow$ non-degenerate, and $\not\Leftarrow$ in general. A positive definite hermitian form is said to be a *(unitary) inner product*. A representation W (of $\mathfrak{sl}(2, \mathbb{R})$) is *unitary* if there exists a unitary inner product $\langle\ ,\ \rangle$ such that

$$\langle \pi(g)u, \pi(g)v\rangle = \langle u, v\rangle \text{ for } \forall u, v \in W,\ \forall g \in G = SL(2, \mathbb{R}).$$

Translate the unitarity condition into Lie algebras:
Take an element $A \in \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, put $g = I + \varepsilon A$ ($\overline{\varepsilon} = \varepsilon$). Then

$$\begin{aligned} \langle \pi(g)u, \pi(g)v\rangle &\equiv \langle u + \varepsilon\pi'(A)u, v + \varepsilon\pi'(A)v\rangle \mod O(\varepsilon^2) \\ &\equiv \langle u, v\rangle + \varepsilon(\langle \pi'(A)u, v\rangle + \langle u, \pi'(A)v\rangle) \mod O(\varepsilon^2). \end{aligned}$$

Thus, since $\langle u, v\rangle = \langle \pi(g)u, \pi(g)v\rangle$, we have

$$\langle \pi'(A)u, v\rangle + \langle u, \pi'(A)v\rangle = 0 \text{ for } \forall u, v \in W \text{ and } \forall A \in \mathfrak{g}. \tag{4.1}$$

4.2 Unitary representation of $\mathfrak{su}(1,1)$

We define

$$\begin{aligned}
\mathfrak{su}(1,1) &:= \mathfrak{u}(1,1)\bigcap \mathfrak{sl}(2,\mathbb{C}) \qquad \text{Example 2.3 (3) and Example 2.1} \\
&= \left\{\begin{pmatrix} i\alpha & \beta+i\gamma \\ \beta-i\gamma & i\delta \end{pmatrix} \mid \alpha,\beta,\gamma,\delta\in\mathbb{R}\right\}\bigcap\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{array}{c} a,b,c,d\in\mathbb{C}, \\ a+d=0 \end{array}\right\} \\
&= \left\{\begin{pmatrix} i\alpha & \beta+i\gamma \\ \beta-i\gamma & -i\alpha \end{pmatrix} \mid \alpha,\beta,\gamma\in\mathbb{R}\right\}.
\end{aligned}$$

Note that $\begin{pmatrix} i\alpha & \beta+i\gamma \\ \beta-i\gamma & -i\alpha \end{pmatrix} = \alpha iH+\beta(X+Y)+\gamma i(X-Y)$, and $\{iH, X+Y, i(X-Y)\}$ is a basis of $\mathfrak{su}(1,1)$.

Suppose $A = iH$. Then the condition (4.1) is

$$\begin{aligned}
0 &= \langle \pi'(iH)u, v\rangle + \langle u, \pi'(iH)v\rangle \\
&= \langle i\pi'(H)u, v\rangle + \langle u, i\pi'(H)v\rangle \\
&= i\langle \pi'(H)u, v\rangle - i\langle u, \pi'(H)v\rangle,
\end{aligned}$$

and thus

$$\langle \pi'(H)u, v\rangle = \langle u, \pi'(H)v\rangle. \tag{4.2}$$

Note that π' is complex linear, i.e.,

$$\pi'(\alpha A) = \alpha\pi'(A) \text{ for } \forall\alpha\in\mathbb{C},\ \forall A\in\mathfrak{su}(1,1).$$

Exercise 4.3. *Other two conditions for $A = X+Y$ and $A = i(X-Y)$ can be written as*

$$\langle \pi'(X)u, v\rangle + \langle u, \pi'(Y)v\rangle = 0. \tag{4.3}$$

4.3 H-invariance

Lemma 4.4. *Suppose $W = V(\lambda, I)$ has a non-degenerate hermitian form $\langle\ ,\ \rangle$ such that*

$$\langle \pi'(H)u, v\rangle = \langle u, \pi'(H)v\rangle \text{ for } \forall u,v\in W.$$

Then $I\subset\mathbb{R}$, $\langle v_\mu, v_\mu\rangle \neq 0$ for $\forall\mu\in I$, and $\langle v_\mu, v_{\mu'}\rangle = 0$ for $\forall\mu,\mu'\in I$ with $\mu\neq\mu'$.

Proof. Let $\mu, \mu' \in I$. From

$$\begin{aligned}\langle \pi'(H)v_\mu, v_{\mu'}\rangle &= \langle \mu v_\mu, v_{\mu'}\rangle = \mu\langle v_\mu, v_{\mu'}\rangle, \\ \langle v_\mu, \pi'(H)v_{\mu'}\rangle &= \langle v_\mu, \mu' v_{\mu'}\rangle = \overline{\mu'}\langle v_\mu, v_{\mu'}\rangle\end{aligned}$$

and (4.2), we have

$$(\mu - \overline{\mu'})\langle v_\mu, v_{\mu'}\rangle = 0. \tag{4.4}$$

Since $\langle\ ,\ \rangle$ is non-degenerate on $V(\lambda, I)$,

$$I \ni \forall\mu \neq 0,\ \exists\mu' \in I \text{ such that } \langle \mu, \mu'\rangle \neq 0.$$

Hence,

$$I \ni \forall\mu \neq 0,\ \exists\mu' \in I \text{ such that } \mu = \mu'.$$

In particular, since $\mu' \in \mu + 2\mathbb{Z}$,

$$\operatorname{Im}\mu = \operatorname{Im}\overline{\mu'} = -\operatorname{Im}\mu' = -\operatorname{Im}\mu.$$

This shows $\mu \in \mathbb{R}$ for all $\mu \in I$. Moreover, if $\mu, \mu' \in I \subset \mathbb{R}$ such that $\mu \neq \mu'$, then (4.4) implies that $\langle v_\mu, v_{\mu'}\rangle = 0$. Since $\langle\ ,\ \rangle$ is non-degenerate, $\langle v_\mu, v_\mu\rangle \neq 0$. □

From now on, we may and will assume $I \subset \mu_0 + 2\mathbb{Z} \subset \mathbb{R}$.

4.4 X,Y condition

Suppose $W = V(\lambda, I)$ is a unitary representation with an inner product $\langle\ ,\ \rangle$. We examine the condition (4.3):

$$\langle \pi'(X)u, v\rangle + \langle u, \pi'(Y)v\rangle = 0 \text{ for } \forall u, v \in W.$$

If $\mu, \mu + 2 \in I$, then

$$\begin{aligned} 0 &= \langle \pi'(X)v_\mu, v_{\mu+2}\rangle + \langle v_\mu, \pi'(Y)v_{\mu+2}\rangle \\ &= \langle \frac{\mu - \lambda}{2} v_{\mu+2}, v_{\mu+2}\rangle + \langle v_\mu, \frac{-\mu - 2 - \lambda}{2} v_\mu\rangle \\ &= \frac{\mu - \lambda}{2}\langle v_{\mu+2}, v_{\mu+2}\rangle - \frac{\mu + 2 + \overline{\lambda}}{2}\langle v_\mu, v_\mu\rangle. \end{aligned} \tag{4.5}$$

Thus

$$\begin{aligned} \frac{\langle v_{\mu+2}, v_{\mu+2}\rangle}{\langle v_\mu, v_\mu\rangle}(\mu - \lambda)(\mu + 2 + \lambda) &= (\mu + 2 + \overline{\lambda})(\mu + 2 + \lambda) \\ &= |\mu + 2 + \lambda|^2 \geq 0. \end{aligned}$$

If $\mu, \mu+2 \in I$, then $(\mu-\lambda)(\mu+2+\lambda) \in \mathbb{R}_{\geq 0}$. Since $(\mu-\lambda)(\mu+2+\lambda) = (\mu+1)^2 - (\lambda+1)^2$, we have

$$(\lambda+1)^2 \in (\mu+1)^2 + \mathbb{R}_{\geq 0} \subset \mathbb{R}. \tag{4.6}$$

So, $\lambda+1 \in \mathbb{R}$ or $\lambda+1 \in \sqrt{-1}\mathbb{R}$.

4.5 The case (1) in Theorem 3.27

We consider $V(\lambda, I)$ with the case $I = \mu_0 + 2\mathbb{Z}$, $\lambda \in \mathbb{C}$ with $\pm\lambda \notin I$.

4.5.1 The case $\lambda + 1 \in \sqrt{-1}\mathbb{R}$

If $\lambda + 1 \neq 0$, then $\pm\lambda \notin I$ is automatic, since $\lambda \notin \mathbb{R}$.

If $\lambda + 1 = 0$, then the condition $\pm\lambda \notin I$ exclude the case

$$I = 1 + 2\mathbb{Z} = \{\text{odd integers}\}.$$

Note that for all $\mu \in I$, we have

$$\frac{\mu+2+\overline{\lambda}}{2} = \frac{\mu+1+\overline{1+\lambda}}{2} = \frac{\mu+1-(1+\lambda)}{2} = \mu - \lambda \neq 0.$$

This and (4.5) imply that

$$\langle v_{\mu+2}, v_{\mu+2} \rangle = \langle v_\mu, v_\mu \rangle \text{ for all } \mu \in I.$$

This show that $\{v_\mu \mid \mu \in \mu_0 + 2\mathbb{Z}\}$ is an orthonormal basis basis of $V(\lambda, \mu_0 + 2\mathbb{Z})$ with respect to the inner product $\langle\ ,\ \rangle$. This class of unitary representations $V(\lambda, \mu_0 + 2\mathbb{Z})$, $\lambda + 1 \in \sqrt{-1}\mathbb{R}$, $\mu_0 + 2\mathbb{Z} \subset \mathbb{R}$ such that

$$(\lambda, \mu_0 + 2\mathbb{Z}) \neq (-1, \{\text{odd integers}\})$$

is called (unitary) *principal series representations*.

4.5.2 The case $\lambda + 1 \in \mathbb{R}$, $\lambda + 1 \neq 0$

The necessary condition (4.6):

$$(\lambda+1)^2 \leq (\mu+1)^2 \quad \text{for all } \mu \in I. \tag{4.7}$$

We may and will take μ_0 with $-2 < \mu_0 \leq 0$ as a representative of $I = \mu_0 + 2\mathbb{Z}$. Note that

$$(\mu_0+1)^2 = \min\{(\mu+1)^2 \mid \mu \in I\}.$$

Then the condition (4.7) is equivalent to

$$(\lambda+1)^2 \leq (\mu_0+1)^2,$$

that is,

$$-|\mu_0+1| \leq \lambda+1 \leq |\mu_0+1|.$$

The irreducibility condition $\pm\lambda \notin \mu_0 + 2\mathbb{Z}$ implies that

$$\pm(\lambda+1) \notin (\mu_0+1) + 2\mathbb{Z},$$

that is,

$$-|\mu_0+1| < \lambda+1 < |\mu_0+1|.$$

This class of unitary representations $V(\lambda, \mu_0 + 2\mathbb{Z})$, $-2 < \mu_0 \leq 0$, $\lambda \in \mathbb{R}$, $0 < |\lambda+1| < |\mu_0+1|$ is called *complementary series representations.*

4.6 The case (2) (and (3)) in Theorem 3.27

We consider $V(\lambda, I)$ with $I = \lambda + 2\mathbb{Z}_{\leq 0} \subset \mathbb{R}$, $\lambda \notin \mathbb{Z}_{\geq 0}$. We apply (4.6) for $\mu = \lambda - 2$. Then we obtain $\lambda \leq 0$. The condition $\lambda \notin \mathbb{Z}_{\geq 0}$ implies $\lambda < 0$. For the case (3), we consider $V(\lambda, I)$ with

$$I = -\lambda + 2\mathbb{Z}_{\geq 0} \subset \mathbb{R},\ \lambda \notin \mathbb{Z}_{\geq 0}.$$

We apply (4.6) for $\mu = -\lambda$. Then we obtain $\lambda \leq 0$. The condition $\lambda \notin \mathbb{Z}_{\geq 0}$ implies $\lambda < 0$. These classes of representations $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$ and $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$ with $\lambda < 0$ are called *discrete series representations.*

4.7 The case (4) in Theorem 3.27

We consider $V(\lambda, I)$ with $I = [-\lambda, \lambda]$, $\lambda \in \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathbb{Z}_{>0}$, we apple (4.6) for $\mu = \lambda - 2$. Then we obtain $\lambda \leq 0$, which is a contradiction. Then the case $\lambda < 0$ is not unitary. The representation $V(0, \{0\}) = \mathbb{C}$ is called a *trivial representation.*

4.8 Irreducible unitary representations

As a summary:

Theorem 4.5. *The list of irreducible unitary representations of* $\mathfrak{su}(1,1)$ *of the form* $V(\lambda, I)$ *is*

(1) $V(\lambda, \mu_0 + 2\mathbb{Z})$; $\lambda+1 \in \sqrt{-1}\mathbb{R}$, $\mu_0 + 2\mathbb{Z} \subset \mathbb{R}$, $(\lambda, \mu_0 + 2\mathbb{Z}) \neq (-1, \{odd\})$,

(1′) $V(\lambda, \mu_0 + 2\mathbb{Z})$; $-2 < \mu_0 \leq 0$, $\lambda \in \mathbb{R}$, $0 < |\lambda + 1| < |\mu_0 + 1|$,

(2) $V(\lambda, \lambda + 2\mathbb{Z}_{\leq 0})$; $\lambda < 0$,

(3) $V(\lambda, -\lambda + 2\mathbb{Z}_{\geq 0})$; $\lambda < 0$,

(4) $V(0, \{0\}) = \mathbb{C}$.

irreducible	$\lambda + 1$: pure imaginary	$\lambda + 1$: real
(1)	(1): principal series	(1′): complementary series
(2),(3)	no	(2),(3): discrete series
(4)	no	(4): trivial

Remark 4.6. *In order to obtain the complete list of irreducible unitary representations of* $\mathfrak{su}(1,1)$*, we need the following:*

- *Every irreducible unitary representation does arise as a subrepresentation of some* $V(\lambda, \mu_0 + 2\mathbb{Z})$.
- *We only discuss the necessary conditions to be unitary. We need to show that every representations above are actually unitary.*
- *We have not introduced the notion of isomorphism of representations. There are a few nontrivial isomorphism between the representations above. For example,*

  $$V(\lambda, \mu_0 + 2\mathbb{Z}) \cong V(-\lambda - 2, \mu_0 + 2\mathbb{Z}).$$

  To obtain the complete list, we should exclude such duplications.

These matters are omitted in this note.